Math 10860, Honors Calculus 2

Homework 8 **NAME**:

Solutions

1. **HELD OVER FROM HOMEWORK 7; TO BE TURNED IN**: Remember that there are no silver-bullet rules for substitution. Just try to substitute for an expression that appears frequently or prominently. If two different troublesome expressions appear, try to express them both in terms of some new expression. **Do any *two* of these**.

   (a)
$$\int \frac{dx}{\sqrt{1+e^x}}.$$

Solution: Try $u=\sqrt{1+e^x}$, so $du = e^x dx/(2\sqrt{1+e^x})$, so $dx/\sqrt{1+e^x} = 2du/e^x = 2du/(u^2-1)$. We get

$$\begin{aligned}
\int \frac{dx}{\sqrt{1+e^x}} &= \int \frac{2du}{u^2-1} \\
&= \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \\
&= \log|u-1| - \log|u+1| \\
&= \log\left|\frac{u-1}{u+1}\right| \\
&= \log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|.
\end{aligned}$$

But in fact the absolute value signs are not needed here: $\sqrt{1+e^x} \geq 1$ always, so we are never attempting to evaluate log at a negative argument, and we can write

$$\int \frac{dx}{\sqrt{1+e^x}} = \log\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right).$$

Note: You might try to be fancy, and quote

$$\int \frac{du}{u^2-1} = -\tanh^{-1} u$$

(this is not a standard integral, but it appears in some lists on the backs of calculus books). But this is not exactly correct; $\tanh^{-1}$ (inverse hyperbolic tangent) is

defined only on $(-1,1)$, so the above identity is true only on that domain. And for this problem, u is always > 1.
It happens that the inverse hyperbolic cotangent function $\coth^{-1}$ is defined exactly where $\tanh^{-1}$ is not defined — on $(-\infty, 1) \cup (1, \infty)$, and on its domain satisfies

$$\int \frac{du}{u^2 - 1} = -\coth^{-1} u.$$

So one can make this approach correct by saying

$$\int \sqrt{1+e^x}dx = -2\coth^{-1}\sqrt{1+e^x}.$$

(Technically, to be fully correct one needs to say that $\coth^{-1}$ should have its domain restricted to positive inputs).

(b)

$$\int \frac{4^x+1}{2^x+1}\, dx.$$

Solution: Make the substitution $u = 2^x + 1 = e^{x\log 2} + 1$. We have

$$du = \log 2 e^{x\log 2}\, dx$$

so

$$dx = \frac{1}{(\log 2)}\frac{1}{2^x}\, du = \frac{1}{(\log 2)}\frac{1}{u-1}\, du,$$

and

$$4^x + 1 = (2^x)^2 + 1 = (u-1)^2 + 1.$$

We get

$$\begin{aligned}
\int \frac{4^x+1}{2^x+1}\, dx &= \frac{1}{\log 2}\int \frac{(u-1)^2+1}{u(u-1)}\, du \\
&= \frac{1}{\log 2}\int \left(1 - \frac{1}{u} + \frac{1}{u(u-1)}\right)\, du \\
&= \frac{1}{\log 2}\int \left(1 - \frac{1}{u} + \frac{1}{u-1} - \frac{1}{u}\right)\, du \\
&= \frac{1}{\log 2}\int \left(1 - \frac{2}{u} + \frac{1}{u-1}\right)\, du \\
&= \frac{1}{\log 2}\left(u - 2\log|u| + \log|u-1|\right) \\
&= \frac{1}{\log 2}\left((2^x+1) - 2\log(2^x+1) + \log 2^x\right) \\
&= \frac{1}{\log 2}\left((2^x+1) - 2\log(2^x+1) + x\log 2\right) \quad \text{(alternate form).}
\end{aligned}$$

(c)

$$\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\ dx.$$

Solution: Note that x is restricted here to be in the set $(-\infty,-1]\cup[1,\infty)$. Start with $u=1/x$ (so u is restricted to $[-1,1]\setminus\{0\}$), so $du=-dx/x^2$, so $-du=dx/x^2$. Also

$$\sqrt{\frac{x-1}{x+1}}=\sqrt{\frac{1-1/x}{1+1/x}}=\sqrt{\frac{1-u}{1+u}},$$

so

$$\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\ dx=-\int\sqrt{\frac{1-u}{1+u}}\ du.$$

We can manipulate the integrand to make it amenable to a trigonometric substitution:

$$\int\sqrt{\frac{1-u}{1+u}}\ du=\int\sqrt{\frac{1-u}{1+u}}\frac{\sqrt{1-u}}{\sqrt{1-u}}\ du=\int\frac{1-u}{\sqrt{1-u^2}}\ du.$$

Via substitution $u=\sin t$ (which is valid — u is restricted to $[-1,1]$), $du=\cos t\ dt$, we get

$$\int\frac{1-u}{\sqrt{1-u^2}}\ du=\int\frac{1-\sin t}{\cos t}\ \cos t\ dt=\int(1-\sin t)\ dt=t+\cos t.$$

So

$$\begin{aligned}\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\ dx &= -t-\cos t\\ &= -\arcsin u-\cos(\arcsin u)\\ &= -\arcsin(1/x)-\cos(\arcsin(1/x)).\end{aligned}$$

If $x\geq 1$, then by the standard right-triangle argument, $\cos(\arcsin(1/x))=\sqrt{1-1/x^2}$, and this is also valid if $x\leq -1$, so

$$\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\ dx=-\arcsin(1/x)-\sqrt{1-1/x^2}.$$

2. Next, an integral where it might not be too ridiculous to consider the last resort substitution $t=\tan(x/2)$.

$$\int\frac{dx}{a\sin x+b\cos x}.\quad (a,b\text{ arbitrary constants}).$$

Solution: Via $t=\tan(x/2)$ we get

$$\begin{aligned}\int\frac{dx}{a\sin x+b\cos x} &= \int\left(\frac{1}{\frac{2at}{1+t^2}+\frac{b(1-t^2)}{1+t^2}}\right)\frac{2dt}{1+t^2}\\ &= \int\frac{2dt}{b+2at-bt^2}.\end{aligned}$$

We deal first with a boundary cases of this integral. If $b = 0$, then the integral becomes

$$\frac{1}{a}\int \frac{dt}{t} = \frac{\log|t|}{a} = \frac{\log|\tan(x/2)|}{a}.$$

(In this case the integral is $(1/a)\int \csc x\ dx$, which is more traditionally presented as $\log|\csc x - \cot x|$; the two answers are easily checked to be the same).

Otherwise, whatever the values of a, b, the quadratic in the denominator always has two real roots (its discriminant (the "$b^2 - 4ac$" in the quadratic formula) is $4a^2 + 4b^2$, which is always positive). So the denominator factors into two real roots. Specifically:

$$b + 2at - bt^2 = -b\left(t - \frac{a + \sqrt{a^2 + b^2}}{b}\right)\left(t - \frac{a - \sqrt{a^2 + b^2}}{b}\right).$$

A partial fractions decomposition gives:

$$\frac{2}{b + 2at - bt^2} = \frac{1}{\sqrt{a^2 + b^2}}\left(\frac{1}{t - \frac{a-\sqrt{a^2+b^2}}{b}} - \frac{1}{t - \frac{a+\sqrt{a^2+b^2}}{b}}\right).$$

The value of the integral when $b \neq 0$ is thus (obviously):

$$\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2 + b^2}}\log\left|\frac{\tan(x/2) - \frac{a-\sqrt{a^2+b^2}}{b}}{\tan(x/2) - \frac{a+\sqrt{a^2+b^2}}{b}}\right|.$$

3. Next, some integrands appropriate for partial fractions. **Do any *one* of these**.

(a)

$$\int \frac{2x^2 + 7x - 1}{x^3 - 3x^2 + 3x - 1}\ dx.$$

Solution: Have

$$x^3 - 3x^2 + 3x - 1 = (x - 1)^3$$

and

$$\frac{2x^2 + 7x - 1}{x^3 - 3x^2 + 3x - 1} = \frac{8}{(x - 1)^3} + \frac{11}{(x - 1)^2} + \frac{2}{x - 1},$$

so

$$\int \frac{2x^2 + 7x - 1}{x^3 - 3x^2 + 3x - 1}\ dx = -\frac{4}{(x - 1)^2} - \frac{11}{(x - 1)} + 2\log|x - 1|.$$

(b)

$$\int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1}\ dx.$$

Have

$$x^3 + 2x^2 + 2x + 1 = (x + 1)(x^2 + x + 1)$$

and

$$\frac{3x^2+3x+1}{x^3+2x^2+2x+1} = \frac{2x}{x^2+x+1} + \frac{1}{x+1}.$$

Now

$$\begin{aligned} \frac{2x}{x^2+x+1} &= \frac{2x}{(x+(1/2))^2+(\sqrt{3}/2)^2} \\ &= \frac{2(x+(1/2))}{(x+(1/2))^2+(\sqrt{3}/2)^2} - \frac{1}{(x+(1/2))^2+(\sqrt{3}/2)^2}. \end{aligned}$$

Each of the three terms

$$\frac{1}{x+1}, \quad \frac{2(x+(1/2))}{(x+(1/2))^2+(\sqrt{3}/2)^2}, \quad \frac{1}{(x+(1/2))^2+(\sqrt{3}/2)^2}$$

are readily integrable, and

$$\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx = \log|x+1| + \log|(x+(1/2))^2+(\sqrt{3}/2)^2| + \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right).$$

(For the last integral:

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$$

is easily derived from $\int dx/(x^2+1) = \tan^{-1}x$.)

(c)

$$\int \frac{3x}{(x^2+x+1)^3}\,dx.$$

The integral is already in the correct form for partial fractions. We write

$$\frac{3x}{(x^2+x+1)^3} = \frac{3(x+(1/2))}{(x+(1/2))^2+(\sqrt{3}/2)^2)^3} - \frac{3/2}{(x+(1/2))^2+(\sqrt{3}/2)^2)^3},$$

and use reduction formulae from class to get:

$$\int \frac{3x}{(x^2+x+1)^3}\,dx = -\left(\frac{2x^3+3x^2+4x+3}{2(x^2+x+1)^2}\right) + \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right).$$

4. Next, a pot-pourri with a (slightly non-obvious) trigonometric flavor. **Do part (a) and *one* of the other two**.

(a)

$$\int \sqrt{1-4x-2x^2}\,dx.$$

First complete the square:

$$1-4x-2x^2 = -2(x^2+2x-1/2) = -2((x+1)^2-3/2) = 3-2(x+1)^2 = (\sqrt{3})^2-(\sqrt{2}(x+1)^2).$$

Now make substitution

$$\frac{\sqrt{2}}{\sqrt{3}}(x+1) = \sin t.$$

So

$$\sqrt{(\sqrt{3})^2 - (\sqrt{2}(x+1)^2)} = \sqrt{3}\sqrt{1-\sin^2 t} = \sqrt{3}\cos t,$$

and

$$dx = \frac{\sqrt{3}}{\sqrt{2}}\cos t\ dt,$$

so

$$\int \sqrt{1-4x-2x^2}\ dx = \frac{3}{\sqrt{2}}\int \cos^2 t\ dt = \frac{3}{2\sqrt{2}}\left(t + \frac{\sin 2t}{2}\right).$$

Now

$$\frac{\sin 2t}{2} = \sin t\cos t = \frac{\sqrt{2}}{\sqrt{3}}(x+1)\cos t.$$

If we are in the regime where $x+1 \geq 0$, so t is an angle between 0 and $\pi/2$, then from a right-angled triangle we get

$$\cos t = \frac{\sqrt{1-4x-2x^2}}{\sqrt{3}}.$$

So, for $x+1 \geq 0$,

$$\begin{aligned}
\int \sqrt{1-4x-2x^2}\ dx &= \frac{3}{2\sqrt{2}}\left(\arcsin\left(\frac{\sqrt{2}(x+1)}{\sqrt{3}}\right) + \frac{\sqrt{2}}{\sqrt{3}}(x+1)\frac{\sqrt{1-4x-2x^2}}{\sqrt{3}}\right) \\
&= \frac{3}{2\sqrt{2}}\arcsin\left(\frac{\sqrt{2}(x+1)}{\sqrt{3}}\right) + \frac{(x+1)}{2}\sqrt{1-4x-2x^2}.
\end{aligned}$$

This also works when $x+1 \leq 0$, so is the final answer.

(b)

$$\int \cos x\sqrt{9+25\sin^2 x}\ dx.$$

A natural substitution is $\sin x = \frac{3}{5}\tan t$. Then

$$\sqrt{9+25\sin^2 x} = 3\sqrt{1+\tan^2 t} = 3\sec t,$$

and

$$\cos x\ dx = \frac{3}{5}\sec^2 t\ dt,$$

so

$$\begin{aligned}
\int \cos x\sqrt{9+25\sin^2 x}\ dx &= \frac{9}{5}\int \sec^3 t\ dt \\
&= \frac{9}{10}\left(\sec t\tan t + \log|\sec t + \tan t|\right)
\end{aligned}$$

(using a previous result on $\int \sec^3 w\ dw$).

Assuming we are in the regime where t is an angle between 0 and $\pi/2$, since we have

$$\tan t = \frac{5\sin x}{3},$$

so from a right-angled triangle we get that

$$\sec t = \frac{\sqrt{9+25\sin^2 x}}{3}$$

and so

$$\int \cos x\sqrt{9+25\sin^2 x}\ dx = \frac{9}{10}\left(\frac{5\sin x\sqrt{9+25\sin^2 x}}{9} + \log\left|\frac{\sqrt{9+25\sin^2 x}}{3} + \frac{5\sin x}{3}\right|\right).$$

Since there is no potential complication with this function in terms of differentiating in other regimes of x, this is the answer for all x.

(c)

$$\int e^{4x}\sqrt{1+e^{2x}}\ dx.$$

Start with $u = 1+e^{2x}$, so $du = 2e^{2x}dx$, and $dx = du/(2(u-1))$. Also $(u-1)^2 = e^{4x}$. We get

$$\begin{aligned}
\int e^{4x}\sqrt{1+e^{2x}}\ dx &= \frac{1}{2}\int (u-1)\sqrt{u}\ du \\
&= \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3} \\
&= \frac{(1+e^{2x})^{5/2}}{5} - \frac{(1+e^{2x})^{3/2}}{3}.
\end{aligned}$$

5. Finally, another pot-pourri. Who knows what methods might be needed? **Do any *two* of these**.

(a)

$$\int \frac{x\arctan x}{(1+x^2)^3}\ dx.$$

Via integration by parts, with

$$u = \arctan x, \qquad du = \frac{dx}{1+x^2},$$

$$dv = \frac{x}{(1+x^2)^3}dx, \qquad v = \frac{-1}{4(1+x^2)^2},$$

get

$$\int \frac{x \arctan x}{(1+x^2)^3}\, dx = \frac{-\arctan x}{4(1+x^2)^2} + \int \frac{dx}{4(1+x^2)^3}$$
$$= \frac{-\arctan x}{4(1+x^2)^2} + \frac{1}{32}\left(\frac{x(5+3x^2)}{(1+x^2)^2} + 3\arctan x\right).$$

(This last integral can be obtained from a reduction formula that we derived in class).

(b)

$$\int \log\sqrt{1+x^2}\, dx.$$

Via integration by parts, with

$$u = \log\sqrt{1+x^2}, \qquad du = \frac{xdx}{1+x^2},$$

$$dv = dx, \qquad v = x,$$

get

$$\int \log\sqrt{1+x^2}\, dx = x\log\sqrt{1+x^2} - \int \frac{x^2 dx}{1+x^2}$$
$$= x\log\sqrt{1+x^2} - \int\left(1 - \frac{1}{1+x^2}\right)\, dx$$
$$= x\log\sqrt{1+x^2} - x + \arctan x.$$

(c)

$$\int \sqrt{\tan x}\, dx.$$

Solution: Substituting $u = \sqrt{\tan x}$,

$$du = \frac{\sec^2 x\ dx}{2\sqrt{\tan x}} = \frac{(\tan^2 x + 1)\ dx}{2u} = \frac{u^4 + 1)\ dx}{2u},$$

so

$$dx = \frac{2u}{u^4 - 1}\, du,$$

and

$$\int \sqrt{\tan x}\, dx = \int \frac{2u^2}{u^4-1}\, du.$$

Factoring $u^4 - 1 = (u^2 + \sqrt{2}u + 1)(u^2 - \sqrt{2}u + 1)$, and writing

$$\frac{2u}{u^4-1} = -\frac{\sqrt{2}}{2}\left(\frac{u}{u^2+\sqrt{2}u+1} - \frac{u}{u^2-\sqrt{2}u+1}\right),$$

after some (lots of) painful algebra, we get to

$$\begin{aligned}\int \sqrt{\tan x}\, dx &= -\frac{\sqrt{2}}{4}\log\left|\tan x + \sqrt{2\tan x} + 1\right| \\ &\quad +\frac{\sqrt{2}}{4}\log\left|\tan x - \sqrt{2\tan x} + 1\right| \\ &\quad +\frac{\sqrt{2}}{2}\arctan\left(\sqrt{2\tan x} + 1\right) \\ &\quad -\frac{\sqrt{2}}{2}\arctan\left(-\sqrt{2\tan x} + 1\right)\end{aligned}$$

(obviously).

6. This question concerns the function f defined by $f(x) = \sqrt{x}$, and its Taylor polynomial of degree 3 at $a = 4$ (i.e., $P_{3,4,f}(x)$).

   (a) Find $P_{3,4,f}(x)$.

   Solution: We have

$$f(x) = \sqrt{x} \qquad f(4) = 2$$

$$f'(x) = \tfrac{1}{2\sqrt{x}} \qquad f'(4) = \tfrac{1}{4}$$

$$f''(x) = \tfrac{-1}{4}x^{-3/2} \qquad f''(4) = \tfrac{-1}{32}$$

$$f'''(x) = \tfrac{3}{8}x^{-5/2} \qquad f'''(4) = \tfrac{3}{256}$$

$$f''''(x) = \tfrac{-15}{16}x^{-7/2} \quad \text{(not needed for } P_{3,4,f}(x)\text{, but needed for } R_{3,4,f}(x)\text{).}$$

   So

$$P_{3,4,f}(x) = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}.$$

   (b) Write $R_{3,4,f}(x)$ in both the integral form and the Lagrange form.

   Solution:

$$\text{Integral form:} \quad \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt = \frac{-5}{32}\int_4^x (x-t)^3 t^{-7/2}dt.$$

$$\text{Lagrange form:} \quad \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} = \frac{-5c^{-7/2}(x-4)^4}{128},$$

   where c is some number between a and x.

(c) Use Taylor's theorem (and the computations of the previous parts) to show that

$$\sqrt{5} = \frac{36640 \pm 5}{16384}.$$

(This is a true story: a calculator suggests that $16384\sqrt{5} = 36635.7\cdots$. The fraction above gives $\sqrt{5}$ correct to 3 decimal places.)

Solution: We take $x = 5$, to get

$$P_{3,4,f}(5) = 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} = \frac{1145}{512} = \frac{36640}{16384}.$$

So, since

$$\sqrt{5} = f(5) = P_{3,4,f}(5) + R_{3,4,f}(5),$$

we would be done if we could show

$$|R_{3,4,f}(5)| \leq \frac{5}{16384}.$$

We consider the Lagrange form of the remainder term, which recall at $x = 5$ is

$$\frac{-5c^{-7/2}}{128},$$

where c is some number between 4 and 5. The largest that $-5/(128c^{7/2})$ can be (in absolute value) in this range of c is at $c = 4$ (smaller denominator makes bigger fraction), at which point it takes the value

$$\frac{5}{16384}$$

(in absolute value), and we are done.

7. (a) Find the Taylor polynomial of degree 4 of $f(x) = x^5 + x^3 + x$, at $a = 1$.

Solution: We have $f(x) = x^5 + x^3 + x$, $f'(x) = 5x^4 + 3x^2 + 1$, $f''(x) = 20x^3 + 6x$, $f'''(x) = 60x^2 + 6$ and $f''''(x) = 120x$, so $f(1) = 3$, $f'(1) = 9$, $f''(1) = 26$, $f'''(1) = 66$ and $f''''(1) = 120$, leading to

$$P_{4,1,f}(x) = 3 + 9(x-1) + \frac{26(x-1)^2}{2} + \frac{66(x-1)^3}{6} + \frac{120(x-1)^4}{24}$$

or

$$P_{4,1,f}(x) = 3 + 9(x-1) + 13(x-1)^2 + 11(x-1)^3 + 5(x-1)^4.$$

(b) Express the polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ as a polynomial in $(x-2)$, using the "start from the highest power, and work down" method described in the notes and the lectures.

Solution: Have

$$a(x-2)^4 = ax^4 - 8ax^3 + 24ax^2 - 32ax + 16a,$$

so

$$p(x) = a(x-2)^4 + (8a+b)x^3 + (-24a+c)x^2 + (32a+d)x + (-16a+e).$$

Have

$$(8a+b)(x-2)^3 = (8a+b)x^3 - 6(8a+b)x^2 + 12(8a+b)x - 8(8a+b),$$

so $p(x) = a(x-2)^4 + (8a+b)(x-2)^3$

$$+(24a+6b+c)x^2 + (-64a-12b+d)x + (48a+8b+e).$$

Have

$$(24a+6b+c)(x-2)^2 = (24a+6b+c)x^2 - 4(24a+6b+c)x + 4(24a+6b+c),$$

so $p(x) = a(x-2)^4 + (8a+b)(x-2)^3 + (24a+6b+c)(x-2)^2$

$$+(32a+12b+4c+d)x + (-48a-16b-4c+e).$$

Finally,

$$(32a+12b+4c+d)(x-2) + 2(32a+12b+4c+d) = (32a+12b+4c+d)x,$$

so

$$\begin{aligned} p(x) &= a(x-2)^4 + (8a+b)(x-2)^3 + (24a+6b+c)(x-2)^2 \\ &\quad +(32a+12b+4c+d)(x-2) + (16a+8b+4c+2d+e). \end{aligned}$$

(Ugh!)

MODIFIED VERSION, MAR 30: Express the polynomial $p(x) = Ax^3 + Bx^2 + Cx + D$ as a polynomial in $(x-2)$, using the "start from the highest power, and work down" method described in the notes and the lectures.

Solution: Working from high powers down, we have

$$\begin{aligned} p(x) &= Ax^3 + Bx^2 + Cx + D \\ &= A(x-2)^3 + 6Ax^2 - 12Ax + 8A + Bx^2 + Cx + D \\ &= A(x-2)^3 + (6A+B)x^2 + (-12A+C)x + (8A+D) \\ &= A(x-2)^3 + (6A+B)(x-2)^2 + 4(6A+B)x - 4(6A+B) \\ &\quad +(-12A+C)x + (8A+D) \\ &= A(x-2)^3 + (6A+B)(x-2)^2 + (12A+4B+C)x + (-16A-4B+D) \\ &= A(x-2)^3 + (6A+B)(x-2)^2 + (12A+4B+C)(x-2) \\ &\quad +2(12A+4B+C) + (-16A-4B+D) \\ &= A(x-2)^3 + (6A+B)(x-2)^2 + (12A+4B+C)(x-2) \\ &\quad (8A+4B+2C+D). \end{aligned}$$

(c) Let f be a polynomial of degree n, let a be any number, and let $P_{n,a,f}$ be the Taylor polynomial of f of degree n about a. Explain why $P_{n,a,f} = f$. (You can be quite brief, but please be precise! This fact follows quickly from a couple of results proved in the lectures, so you just need to briefly say what the right combination is.)

Solution: Because $P_{n,a,f}$ is the Taylor polynomial of f of degree n about a, we know (from a theorem proved in class) that $P_{n,a,f}$ agrees with f to order n at a, that is

$$P_{n,a,f} \sim_{n,a} f.$$

But we also know, from a theorem from class, that if two polynomials of degree at most n agree to order n at a, they are equal. Since $P_{n,a,f}$ and f are both polynomials of degree at most n that agree to order n at a, they must be equal, as claimed.

(d) Express the polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ as a polynomial in $(x-2)$, using the result of part (c). (You should get the same answer as you got for part (b). The computations should be much less messy for this part, though!)

Solution: From part (c), we need only compute $P_{4,2,p}(x)$ — this is a polynomial in $(x-2)$, and equals p.

We have

$$\begin{aligned} p(2) &= 16a + 8b + 4c + 2d + e, \\ p'(x) &= 4ax^3 + 3bx^2 + 2cx + d, \end{aligned}$$

so

$$\begin{aligned} p'(2) &= 32a + 12b + 4c + d, \\ p''(x) &= 12ax^2 + 6bx + 2c, \\ p''(2) &= 48a + 12b + 2c, \\ p'''(x) &= 24ax + 6b, \\ p'''(2) &= 48a + 6b, \\ p''''(x) &= 24a, \\ p''''(2) &= 24a, \end{aligned}$$

so

$$\begin{aligned} p(x) = P_{4,2,p}(x) &= (16a + 8b + 4c + 2d + e) \\ &\quad +(32a + 12b + 4c + d)(x-2) \\ &\quad +(24a + 6b + c)(x-2)^2 \\ &\quad +(8a + b)(x-2)^3 \\ &\quad +a(x-2)^4, \end{aligned}$$

exactly what we got in part (b), but much cleaner.

MODIFIED VERSION, MAR 30: Express the polynomial $p(x) = Ax^3 + Bx^2 + Cx + D$ as a polynomial in $(x-2)$, using the result of part (c). (You should get the same answer as you got for part (b). The computations should be much less messy for this part, though!)

Solution: From part (c), we need only compute $P_{3,2,p}(x)$ — this is a polynomial in $(x-2)$, and equals p.

We have

$$p(2) = 8A + 4B + 2C + D,$$
$$p'(x) = 3Ax^2 + 2BX + C,$$

so

$$p'(2) = 12A + 4B + C,$$
$$p''(x) = 6Ax + 2B,$$
$$p''(2) = 12A + 2B,$$
$$p'''(x) = 6A,$$
$$p'''(2) = 6A,$$

so

$$\begin{aligned} p(x) = P_{3,2,p}(x) \quad = \quad & (8A + 4B + 2C + D) \\ & +(12A + 4B + C)(x-2) \\ & +(6A + B)(x-2)^2 \\ & +A(x-2)^3, \end{aligned}$$

exactly what we got in part (b), but much cleaner.

8. An important Taylor polynomial that we did not discuss much in class is that of $\log x$, at $a = 1$ (we can't choose $a = 0$ here, since 0 is not in the domain of log). Actually, it's nicer to consider the function $\log(1+x)$ at $a = 0$.

   (a) By calculating derivatives, find the Taylor polynomial of degree n of $\log(1+x)$ about $a = 0$.

   Solution: Let $f(x) = \log(1+x)$. We have $f'(x) = 1/(1+x)$, $f''(x) = -1/(1+x)^2$, $f'''(x) = 2/(1+x)^3$, $f^{(4)}(x) = -6/(1+x)^4$, and in general $f^{(n)}(x) = (-1)^{n-1}(n-1)!/(1+x)^n$, so that $f(0) = \log 1 = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -6$, and in general $f^{(n)}(0) = (-1)^{n-1}(n-1)!$, leading to

$$P_{n,0,f}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}.$$

(b) Show that for $-1 < x \leq 1$ the remainder term $R_{n,0,\log(1+\cdot)}(x)$ goes to zero as n goes to infinity. **Hint**: It might be better to avoid the Lagrange or integral forms of the remainder term, instead starting with

$$\log(1+x) = \int_0^x \frac{dt}{1+t}.$$

Solution: Following the hint we have

$$\begin{aligned}
\log(1+x) &= \int_0^x \frac{dt}{1+t} \\
&= \int_0^x \left(1 - t + t^2 - \cdots + (-1)^n t^n + \frac{(-1)^{n+1}t^{n+1}}{1+t}\right) dt \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + (-1)^{n+1} \int_0^x \frac{t^{n+1}dt}{1+t}.
\end{aligned}$$

Since, as established in the last part,

$$P_{n+1,0,f}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1},$$

it follows that

$$R_{n+1,0,f}(x) = (-1)^{n+1} \int_0^x \frac{t^{n+1}dt}{1+t}.$$

If $x = 0$ this remainder term is always 0. If $0 < x \leq 1$ (the easy case) we have

$$|R_{n+1,0,f}(x)| = \int_0^x \frac{t^{n+1}dt}{1+t} \leq \int_0^x t^{n+1}dt = \frac{x^{n+2}}{n+2} \to 0.$$

If $-1 < x < 0$ (the trickier case) we have, using $|\int_a^b f| \leq \int_a^b |f|$ for any f integrable on $[a,b]$ with $a < b$,

$$|R_{n+1,0,f}(x)| \leq \int_x^0 \left|\frac{t^{n+1}}{1+t}\right| dt.$$

Now as t varies from x to 0, $|1+t|$ varies from $x+1$ to 1, and in particular $1/|1+t| \leq 1/(1+x)$ (this is a positive quantity). So

$$|R_{n+1,0,f}(x)| \leq \frac{1}{1+x} \int_x^0 |t|^{n+1}dt = \frac{1}{1+x}\left(\left.\frac{-|t|^{n+2}}{n+2}\right|_{t=x}^{0}\right) = \frac{|x|^{n+2}}{(n+2)(1+x)},$$

using $\int |t|^{n+1} = -|t|^{n+2}/(n+2)$ for $t < 0$. For each fixed $x \in (-1,0)$ we have

$$\frac{|x|^{n+2}}{(n+2)(1+x)} \to 0$$

as $n \to \infty$, and so we are done (note that things go wrong at $x = -1$.)

(c) Use Taylor polynomials, and your analysis of the remainder term, to find a rational number that is within ± 0.1 of $\log 2$.

Solution: We have found that $|R_{n,0,f}(1)| \leq 1/(n+1)$, so at $n = 9$ we get $|R_{9,0,f}(1)| \leq .1$. It follows that $P_{9,0,f}(1)$ is within $\pm .1$ of $\log 2$. We have

$$P_{9,0,f}(1) = 1 - \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{8} + \frac{1}{9} = \frac{1879}{2520} \approx 0.745635$$

(which turns out to differ from $\log 2$ by only about 0.05).

(d) **OPTIONAL**: Show that for $x > 1$ the remainder term $R_{n,0,\log(1+\cdot)}(x)$ *does not* go to zero as n goes to infinity.

Solution: For $x > 1$ we have

$$|R_{n+1,0,f}(x)| = \int_0^x \frac{t^{n+1}dt}{1+t} = \int_0^1 \frac{t^{n+1}dt}{1+t} + \int_1^x \frac{t^{n+1}dt}{1+t}.$$

If it was the case that $R_{n+1,0,f}(x) \to 0$, then it would also be the case that $|R_{n+1,0,f}(x)| \to 0$, and then since both $\int_0^1 \frac{t^{n+1}dt}{1+t}$, $\int_1^x \frac{t^{n+1}dt}{1+t}$ are non-negative, it would also be the case that both these integrals thens to 0 as n grows. We show that the second of then does not, a contradiction. We have, for $x > 1$,

$$\int_1^x \frac{t^{n+1}dt}{1+t} \geq \frac{1}{1+x}\int_1^x t^{n+1}dt \geq \frac{x-1}{1+x},$$

which certainly does not go to 0.

(e) **OPTIONAL**: Nevertheless, use Taylor polynomials (slightly cleverly) to find a rational number that is within ± 0.1 of $\log 3$.

Solution: We have $\log 3 = \log(3/2)2 = \log(3/2) + \log 2$. We estimate $\log 2$ to within $\pm .05$ by using $P_{19,0,f}(1) = \frac{33464927}{46558512}$. We estimate $\log(3/2) = \log(1 + (1/2))$ to within $\pm .05$ by using $P_{2,0,f}(1/2) = \frac{3}{8}$; note that the error here is in absolute value at most $(1/2)^3/3 < 1/20$. Adding these two estimates, the net error is at most ± 0.1, so we get a final estimate of

$$\frac{33464927}{46558512} + \frac{3}{8} = \frac{50924369}{46558512} \approx 1.09377;$$

the actual value starts 1.09861.

Alternatively: $\log 3 = -\log(1/3) = -\log(1 + (-2/3))$. We know that at $x = -2/3$, the absolute remainder term $|R_{n,0,f}(x)|$ is at most

$$\frac{|x|^{n+2}}{(n+2)(1+x)} = \frac{3(2/3)^{n+2}}{(n+2)},$$

which falls below .1 at $n = 4$, so that $-P_{4,0,f}(-2/3)$ is with $\pm .1$ of $\log 3$. We have

$$-P_{4,0,f}(-2/3) = \frac{28}{27} \approx 1.03704.$$

9. **OPTIONAL**: Here is (something of) a generalization of the binomial theorem. Recall that the binomial theorem says that for all natural numbers n, and for all real x,

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

where for natural numbers k and n, $\binom{n}{k} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$.

For an arbitrary real number α, and natural number k, define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}$$

(note that this agrees with $\binom{n}{k}$ when $\alpha = n$).

Let $f_\alpha : (-1,\infty) \to \mathbb{R}$ be defined by $f_\alpha(x) = (1+x)^\alpha$.

(a) Show that the Taylor polynomial of degree n of f_α about 0 is

$$P_{n,0,f_\alpha}(x) = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n,$$

and that the remainder term can be expressed as $R_{n,0,f_\alpha}(x) = \binom{\alpha}{n+1}x^{n+1}(1+t)^{\alpha-n-1}$ for some t between 0 and x.

Solution: We have $f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-(n-1))(1+x)^{\alpha-n}$, and so

$$\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))(1+0)^{\alpha-n}}{n!} = \binom{\alpha}{n}.$$

This establishes the claim about the Taylor polynomial. The remainder term, in Lagrange form, is

$$R_{n,0,f_\alpha}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}x^{n+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+t)^{\alpha-n-1}}{(n+1)!}x^{n+1} = \binom{\alpha}{n+1}x^{n+1}(1+t)^{\alpha-n-1}$$

for some t between 0 and x, as claimed.

(b) The remainder term above is quite difficult to pin down. In some special cases, though, it is reasonable.

i. Show that

$$\binom{-1/2}{n+1} = (-1)^{n+1}\frac{\binom{2n+2}{n+1}}{2^{2n+2}}$$

(this requires level-headed algebraic manipulation. It helps to know what you are aiming for in advance!).

Solution: We have $\binom{-1/2}{n+1} =$

$$
\begin{aligned}
& \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(\frac{-(2n-1)}{2}\right)\left(\frac{-(2n+1)}{2}\right)}{(n+1)n(n-1)\cdots(2)(1)} \\
= & \frac{(-1)^{n+1}}{2^{n+1}}\frac{(2n+1)(2n-1)(2n-3)\cdots(3)(1)}{(n+1)n(n-1)\cdots(2)(1)} \\
= & \frac{(-1)^{n+1}}{2^{2n+2}}\frac{2^{n+1}(n+1)n(n-1)\cdots(2)(1)(2n+1)(2n-1)(2n-3)\cdots(3)(1)}{(n+1)n(n-1)\cdots(2)(1)(n+1)n(n-1)\cdots(2)(1)} \\
= & \frac{(-1)^{n+1}}{2^{2n+2}}\frac{(2n+2)2n(2n-2)\cdots(4)(2)(2n+1)(2n-1)(2n-3)\cdots(3)(1)}{(n+1)n(n-1)\cdots(2)(1)(n+1)n(n-1)\cdots(2)(1)} \\
= & \frac{(-1)^{n+1}}{2^{2n+2}}\frac{(2n+2)!}{(n+1)!(n+1)!} \\
= & \frac{(-1)^{n+1}}{2^{2n+2}}\binom{2n+2}{n+1}.
\end{aligned}
$$

as claimed.

ii. Deduce that for $0 < x < 1$, $R_{n,0,f_{-1/2}}(x) \to 0$ as n grows, and that for $-1/2 < x < 0$, $R_{n,0,f_{-1/2}}(x) \to 0$ as n grows.

Solution: We have

$$|R_{n,0,f_{-1/2}}(x)| = \left|\binom{-1/2}{n+1}\right| |x|^{n+1}|1+t|^{-1/2-n-1}.$$

Using

$$\binom{2n+2}{n+1} \leq (1+1)^{2n+2} = 2^{2n+2}$$

we get

$$\left|\binom{-1/2}{n+1}\right| \leq 1$$

so that

$$|R_{n,0,f_{-1/2}}(x)| \leq \left|\frac{x}{1+t}\right|^{n+1} \frac{1}{\sqrt{|1+t|}}$$

for some t between 0 and x.
For $0 < x < 1$ we have

$$|R_{n,0,f_{-1/2}}(x)| \leq \left(\frac{x}{1+t}\right)^{n+1} \frac{1}{\sqrt{1+t}} \leq x^{n+1}$$

since here $t < 0$.
For each x satisfying $-1/2 < x < 0$ we have, for t between 0 and x, that $1+t > 1/2$, so $|1+t| > |x|$ and so

$$|R_{n,0,f_{-1/2}}(x)| \leq \left|\frac{x}{1+t}\right|^{n+1} \frac{1}{\sqrt{|1+t|}} \leq 2\left|\frac{x}{1+t}\right|^{n+1}$$

which goes to 0 as n grows (here using $|1+t| > |x|$).