Math 10860, Honors Calculus 2

Homework 8 **NAME**:

Due by 11pm Friday April 3

Instructions

As I did towards the end of last semester, from here on I will identify a subset of the homework problems that I consider to be "core", and these are the problems that should be turned in. Other problems labelled optional are, of course, highly recommended!

If you have familiarity with LaTeX, or any other math-oriented word processing software, I strongly encourage you to use it for your homework! As a general rule, typing math helps you focus on brevity and clarity of presentation; but at present, there is also the practicality that it will be much easier to transmit homework (between you and me, between me and the grader) if it starts out in electronic form.

If typing homework is unfeasible, then you should write it **NEATLY**, using plenty of blank space, and scan it to pdf. To make the digital copy legible, consider using pen rather than pencil.

However you produce the homework, you should email it to

`math10860homework at gmail.com`

by the deadline.

Like the last homework, some of this homework is a list of integrals to compute. You could in principle do these problems by entering the integrand into `Mathematica` (or some similar program), noting the result, and then verifying it by differentiation. This is **not** what I'm intending. I want to see you tackle these integrals using techniques of integration. For each integral, you should say clearly what method you are using in each step; other than that, no great explanation is need.

Nota bene: I promise that unless explicitly stated otherwise, all the integrals below *have elementary primitives*. I don't promise that the homework is typo-free, and unfortunately even a tiny typo can turn a do-able integration into an impossible one; so alert me if you think that there is a problem with any of these!

Reading for this homework

Class notes, Sections 13.4 (mostly the latter part, on integrals of powers of sin and cos) and 13.5 (partial fractions), and Chapter 14 up to the bottom of page 360 (Taylor polynomials and Taylor's theorem).

Assignment

1. **HELD OVER FROM HOMEWORK 7; TO BE TURNED IN**: Remember that there are no silver-bullet rules for substitution. Just try to substitute for an expression that appears frequently or prominently. If two different troublesome expressions appear, try to express them both in terms of some new expression. **Do any *two* of these**.

   (a)
   $$\int \frac{dx}{\sqrt{1+e^x}}.$$

   (b)
   $$\int \frac{4^x+1}{2^x+1}\, dx.$$

   (c)
   $$\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\, dx.$$

2. Next, an integral where it might not be too ridiculous to consider the last resort substitution $t = \tan(x/2)$.
   $$\int \frac{dx}{a\sin x + b\cos x}. \quad (a, b \text{ arbitrary constants}).$$

3. Next, some integrands appropriate for partial fractions. **Do any *one* of these**.

   (a)
   $$\int \frac{2x^2+7x-1}{x^3-3x^2+3x-1}\, dx.$$

   (b)
   $$\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\, dx.$$

   (c)
   $$\int \frac{3x}{(x^2+x+1)^3}\, dx.$$

4. Next, a pot-pourri with a (slightly non-obvious) trigonometric flavor. **Do part (a) and *one* of the other two**.

   (a)
   $$\int \sqrt{1-4x-2x^2}\, dx.$$

   (b)
   $$\int \cos x\sqrt{9+25\sin^2 x}\, dx.$$

(c)

$$\int e^{4x}\sqrt{1+e^{2x}}\,dx.$$

5. Finally, another pot-pourri. Who knows what methods might be needed? **Do any *two* of these**.

   (a)
   $$\int \frac{x\arctan x}{(1+x^2)^3}\,dx.$$

   (b)
   $$\int \log\sqrt{1+x^2}\,dx.$$

   (c)
   $$\int \sqrt{\tan x}\,dx.$$

6. This question concerns the function f defined by $f(x)=\sqrt{x}$, and its Taylor polynomial of degree 3 at $a=4$ (i.e., $P_{3,4,f}(x)$).

   (a) Find $P_{3,4,f}(x)$.

   (b) Write $R_{3,4,f}(x)$ in both the integral form and the Lagrange form.

   (c) Use Taylor's theorem (and the computations of the previous parts) to show that
   $$\sqrt{5}=\frac{36640\pm 5}{16384}.$$
   (This is a true story: a calculator suggests that $16384\sqrt{5}=36635.7\cdots$. The fraction above gives $\sqrt{5}$ correct to 3 decimal places.)

7. (a) Find the Taylor polynomial of degree 4 of $f(x)=x^5+x^3+x$, at $a=1$.

   (b) Express the polynomial $p(x)=ax^4+bx^3+cx^2+dx+e$ as a polynomial in $(x-2)$, using the "start from the highest power, and work down" method described in the notes and the lectures.

   MODIFIED VERSION, MAR 30: Express the polynomial $p(x)=Ax^3+Bx^2+Cx+D$ as a polynomial in $(x-2)$, using the "start from the highest power, and work down" method described in the notes and the lectures.

   (c) Let f be a polynomial of degree n, let a be any number, and let $P_{n,a,f}$ be the Taylor polynomial of f of degree n about a. Explain why $P_{n,a,f}=f$. (You can be quite brief, but please be precise! This fact follows quickly from a couple of results proved in the lectures, so you just need to briefly say what the right combination is.)

(d) Express the polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ as a polynomial in $(x-2)$, using the result of part (c). (You should get the same answer as you got for part (b). The computations should be much less messy for this part, though!)

MODIFIED VERSION, MAR 30: Express the polynomial $p(x) = Ax^3 + Bx^2 + Cx + D$ as a polynomial in $(x-2)$, using the result of part (c). (You should get the same answer as you got for part (b). The computations should be much less messy for this part, though!)

8. An important Taylor polynomial that we did not discuss much in class is that of $\log x$, at $a = 1$ (we can't choose $a = 0$ here, since 0 is not in the domain of log). Actually, it's nicer to consider the function $\log(1+x)$ at $a = 0$.

   (a) By calculating derivatives, find the Taylor polynomial of degree n of $\log(1+x)$ about $a = 0$.

   (b) Show that for $-1 < x \leq 1$ the remainder term $R_{n,0,\log(1+\cdot)}(x)$ goes to zero as n goes to infinity. **Hint**: It might be better to avoid the Lagrange or integral forms of the remainder term, instead starting with

$$\log(1+x) = \int_0^x \frac{dt}{1+t}.$$

   (c) Use Taylor polynomials, and your analysis of the remainder term, to find a rational number that is within ± 0.1 of $\log 2$.

   (d) **OPTIONAL**: Show that for $x > 1$ the remainder term $R_{n,0,\log(1+\cdot)}(x)$ *does not* go to zero as n goes to infinity.

   (e) **OPTIONAL**: Nevertheless, use Taylor polynomials (slightly cleverly) to find a rational number that is within ± 0.1 of $\log 3$.

9. **OPTIONAL**: Here is (something of) a generalization of the binomial theorem. Recall that the binomial theorem says that for all natural numbers n, and for all real x,

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

where for natural numbers k and n, $\binom{n}{k} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$.

For an arbitrary real number α, and natural number k, define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}$$

(note that this agrees with $\binom{n}{k}$ when $\alpha = n$).

Let $f_\alpha : (-1,\infty) \to \mathbb{R}$ be defined by $f_\alpha(x) = (1+x)^\alpha$.

(a) Show that the Taylor polynomial of degree n of f_α about 0 is

$$P_{n,0,f_\alpha}(x) = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n,$$

and that the remainder term can be expressed as $R_{n,0,f_\alpha}(x) = \binom{\alpha}{n+1}x^{n+1}(1+t)^{\alpha-n-1}$ for some t between 0 and x.

(b) The remainder term above is quite difficult to pin down. In some special cases, though, it is reasonable.

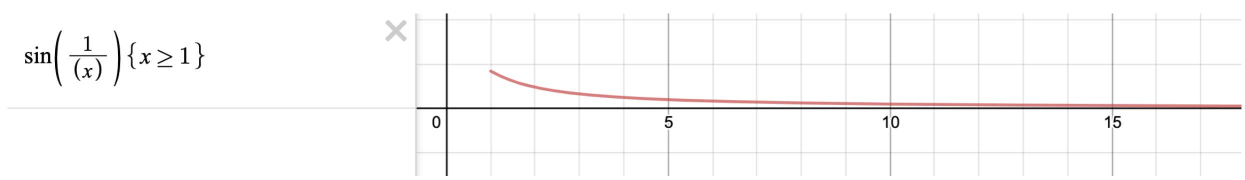
i. Show that

$$\binom{-1/2}{n+1} = (-1)^{n+1}\frac{\binom{2n+2}{n+1}}{2^{2n+2}}$$

(this requires level-headed algebraic manipulation. It helps to know what you are aiming for in advance!).

ii. Deduce that for $0 < x < 1$, $R_{n,0,f_{-1/2}}(x) \to 0$ as n grows, and that for $-1/2 < x < 0$, $R_{n,0,f_{-1/2}}(x) \to 0$ as n grows.

Extra credit problem: Here's a graph of the function $f(x) = \sin(1/x)$ on the interval $[1, \infty)$:



It is positive, and decreasing to 0 as $x \to \infty$. Does

$$\int_1^\infty \sin\left(\frac{1}{x}\right)\,dx$$

converge?