Math 10860, Honors Calculus 2

Homework 9 **NAME**:

Solutions

1. (a) Suppose that f is a function that is defined, and differentiable arbitrarily many times, on some interval I (finite, infinite, whatever). Let a and x be any two numbers in I. Suppose that there is some number $M > 0$ such that $|f^{(n)}(\xi)| \leq M^n$ for every $n \geq 0$ and every $\xi \in [a, x]$. Prove that

$$P_{n,a,f}(x) \to f(x) \quad \text{as } n \to \infty.$$

Solution: We use the Lagrange form of the remainder term (though it should also be possible to use the integral form). We have that there is some number ξ between a and x with

$$\begin{aligned} |R_{n,a,f}(x)| &= \left| \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!} \right| \\ &\leq \frac{(M|x-a|)^{n+1}}{(n+1)!}, \end{aligned}$$

using the bound given on the absolute value of the $(n+1)$st derivative.

Now using the result (proved earlier) that for any $C > 0$

$$\lim_{n\to\infty} \frac{C^n}{n!} = 0,$$

and the fact that for each a and x, $M|x-a|$ is just a fixed positive constant, we get that

$$\frac{(M|x-a|)^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$, and so $|R_{n,a,f}(x)| \to 0$ as $n \to \infty$. This is equivalent to $P_{n,a,f}(x) \to f(x)$ as $n \to \infty$.

(b) We saw (in both notes and lectures) that for every $x \in \mathbb{R}$,

$$P_{n,0,\exp}(x) \to \exp(x) \quad \text{as } n \to \infty.$$

Generalize this: for every base $b > 0$, every $a \in \mathbb{R}$, and every $x \in \mathbb{R}$, the Taylor polynomials of $f(x) = b^x$ centered at a converge to $f(x)$ as n grows. That is,

$$P_{n,a,b^{\cdot}}(x) \to b^x \quad \text{as } n \to \infty.$$

Solution: It's useful to first dismiss the case $b = 1$: here f is the constant function 1, whose Taylor polynomials are all the constant function 1, so clearly the result holds in this case!

Now we deal with the situation where $b \neq 1$. Using part a, we need only show that for each $x \in \mathbb{R}$ there is an $M > 0$ such that for all n, $f^{(n)}(\xi) \leq M^n$ for all ξ between 0 and x, where $f(t) = b^t$.

Now $f(t) = e^{t \log b}$, so for each n

$$f^{(n)}(t) = (\log b)^n e^{t \log b} = (\log b)^n b^t.$$

So, using that f is monotone (either increasing if $b > 1$ or decreasing if $b < 1$) we get that for all ξ between 0 and x,

$$|f^{(n)}(\xi)| \leq |\log b|^n \max\{1, b^x\} \leq |\log b|^n \max\{1, b^x\}^n.$$

If we take $M = |\log b| \max\{1, b^x\}$, then the hypotheses of part (a) are satisfied, and we are done.

2. (a) Use Theorem 15.2 of the course notes (connecting continuity and limits of sequences) to find, for each fixed $a > 0$, $\lim_{n\to\infty} a^{1/n}$.

    Solution: Write $a^{1/n}$ as $e^{(\log a)/n}$. We have $(\log a)/n \to 0$ as $n \to \infty$, and the function $f(x) = e^x$ is continuous at 0, so by Theorem 15.2 we get $\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} f((\log a)/n) = f(0) = 1$.

   (b) Prove a "squeeze theorem" for sequences:

    Let $(a_n), (b_n)$ and (c_n) be sequences with $(a_n), (c_n) \to L$. If eventually (for all $n > n_0$, for some finite n_0) we have $a_n \leq b_n \leq c_n$, then $(b_n) \to L$ also.

    Solution: Fix $\varepsilon > 0$. There is n_1, n_2 such that $n > n_1$ implies $a_n \in (L - \varepsilon, L + \varepsilon)$, and $n > n_2$ implies $c_n \in (L - \varepsilon, L + \varepsilon)$. For $n > \max\{n_0, n_1, n_2\}$ (n_0 as in the statement of the theorem), we have

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

    so $b_n \in (L - \varepsilon, L + \varepsilon)$.

   (c) Use the results of parts (a) and (b) to compute

$$\lim_{n\to\infty} \left(\frac{2n^2 - 1}{3n^2 + n + 2}\right)^{\frac{1}{n}}.$$

    Solution: We have $(2n^2 - 1)/(3n^2 + n + 2) \to 2/3$ as $n \to \infty$, so for all sufficiently large n

$$0.6^{\frac{1}{n}} \leq \left(\frac{2n^2 - 1}{3n^2 + n + 2}\right)^{\frac{1}{n}} \leq 0.7^{\frac{1}{n}}.$$

Since, as we have seen previously, both $0.6^{\frac{1}{n}}, 0.7^{\frac{1}{n}} \to 1$ as $n \to \infty$, the squeeze theorem allows us to conclude

$$\lim_{n\to\infty} \left(\frac{2n^2 - 1}{3n^2 + n + 2} \right)^{\frac{1}{n}} = 1.$$

3. Find the following limits:

(a) $\lim_{n\to\infty} \frac{n}{n+1}$. (For this one, you *must* use the definition of sequence limit).

Solution: We claim that the limit is 1. To show this from the definition, given $\varepsilon > 0$ we need to find n_0 such that $n > n_0$ implies $|n/(n+1) - 1| < \varepsilon$, which is equivalent to $|-1/(n+1)| < \varepsilon$, which is equivalent to $1/(n+1) < \varepsilon$, which is equivalent to $n > (1/\varepsilon) - 1$; so we may take $n_0 = (1/\varepsilon) - 1$.

(b) $\lim_{n\to\infty} \sqrt[n]{n^2 + n}$. (For this and the remaining parts, a soft argument is fine, meaning, you may freely use theorems proven in lectures and/or notes).

Solution: We claim that the limit is 1. We have $\sqrt[n]{n^2 + n} = (n^2 + n)^{1/n} = e^{(\log(n^2+n))/n}$, so (by continuity of the exponential function), it is enough to show that $(\log(n^2 + n))/n \to 0$ as $n \to \infty$, $n \in \mathbb{N}$, for which it suffices to show $(\log(x^2 + x))/x \to 0$ as $x \to \infty$, $x \in \mathbb{R}$, which follows quickly from an application of L'Hôpital's rule.

(c) $\lim_{n\to\infty} \left(\sqrt[8]{n^2 + 1} - \sqrt[4]{n + 1} \right)$.

Solution: The limit is 0. We set $x = \sqrt[8]{n^2 + 1}$ and $y = \sqrt[4]{n + 1} = \sqrt[4]{n^2 + 2n + 1}$, and use

$$x - y = \frac{x^8 - y^8}{x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7}.$$

The numerator is $n^2 + 1 - (n^2 + 2n + 1) = -2n$, and denominator is the sum of terms all of which have the property that, on division by $n^{7/4}$, tend to a constant; for example

$$\frac{x^3y^4}{n^{7/4}} = \frac{(n^2 + 1)^{3/8}(n^2 + 2n + 1)^{4/8}}{n^{7/4}} = \left(1 + \frac{1}{n^2}\right)^{3/8} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)^{4/8} \to 1.$$

So if we divide the original expression through by $n^{7/4}$, the numerator tends to 0 while the denominator tends to a constant, leading to the conclusion that the limit is 0.

(d) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n} \right)$.

Solution: We have $\frac{n}{n+1} - \frac{n+1}{n} = \frac{n^2-(n+1)^2}{n(n+1)} = \frac{-2n+1}{n(n+1)} \to 0$ as $n \to \infty$.

(e) $\lim_{n\to\infty} \frac{2^{n^2}}{n!}$.

Solution: We claim that the limit is infinity. Note that $n! < n^n$ so

$$2^{n^2}/n! > 2^{n^2}/(n^n) = 2^{n^2}/2^{n\log_2 n} = 2^{n^2 - n\log_2 n}.$$

Since $n^2 - n\log_2 n$ goes to infinity with n, so does $2^{n^2-n\log_2 n}$, and so so does $2^{n^2}/n!$.

(f) $\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}\sin(n^n)}{n+1}$.

Solution: In absolute value the nth term is no more than $\sqrt{n}/(n+1)$, which tends to 0 as n grows, so the limit is 0.

4. A *subsequence* of a sequence

$$(a_1, a_2, a_3, \ldots)$$

is a sequence of the form

$$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$$

with $n_1 < n_2 < n_3 \cdots$. In other words, it is a sequence obtained from another sequence by extracting an infinite subset of the elements of the original sequence, *keeping the elements in the same order as they were in the original sequence.*

(a) Consider the sequence

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots\right).$$

For which numbers α is there a subsequence converging to α?

Solution: We claim that there a subsequence converging to α if and only if $\alpha \in [0,1]$.

If $\alpha < 0$ or $\alpha > 1$, there is clearly no subsequence converging to α. If $\alpha = 0$ we may take the subsequence $(1/2, 1/3, 1/4, 1/5, \ldots)$. So now consider $\alpha \in (0,1]$. Because the rationals are dense in $(0,\alpha)$ we can find a sequence of rationals $(r_1, r_2, r_3, \ldots)$ with each $r_i \in (0,\alpha)$ and with $(r_i) \to \alpha$ (simply take r_1 to be any rational in $(0,\alpha)$, r_2 to be any rational in $(\alpha/2, \alpha)$, r_3 to be any rational in $(3\alpha/4, \alpha)$, etc.).

Now note that each rational in $(0,1)$ appears infinitely often in the sequence

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots\right);$$

for example, $1/7$ reappears as $2/14$, $3/21$, etc.. So r_1 can be found somewhere in the sequence, and r_2 can be found somewhere *later* in the sequence, and r_3 can be found somewhere *later still* in the sequence, and so on, giving a subsequence converging to α.

(b) Now consider the same sequence as in part (a), except remove all duplicated terms, so that it begins

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \dots\right).$$

Now for which numbers α is there a subsequence converging to α?

Solution: We can't use the same proof as in part (a), because *no* rational appears more than once in the sequence. However, the result is still the same: there a subsequence converging to α if and only if $\alpha \in [0, 1]$.

As before, trivially there is no subsequence converging to any $\alpha < 0$ or $\alpha > 1$. We get $\alpha = 0$ and $\alpha = 1$ by considering the same subsequences as in part (a).

For $\alpha \in (0, 1)$: Because the rationals are dense, there is a rational number r_1 in the sequence that lies in $(\alpha - \alpha/2, \alpha + \alpha/2)$.

Now we look for a *later* rational in the sequence that lies in $(\alpha - \alpha/4, \alpha + \alpha/4)$. By density we know that there is not just *one* rational number in $(\alpha - \alpha/4, \alpha + \alpha/4)$, there are in fact *infinitely many*. So at least one of those infinitely many must occur in our sequence *after* r_1 occurs; take any one such to be r_2.

By the same argument, we can find a later (later than r_2) rational r_3 in the sequence that lies in $(\alpha - \alpha/8, \alpha + \alpha/8)$. Continuing in this manner, we get a subsequence from our given sequence that converges to α.

5. (a) Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.

Solution: For positive a, $a < \sqrt{2a}$ iff $a^2 < 2a$ iff $a < 2$, and $\sqrt{2a} < 2$ iff $2a < 4$ iff $a < 2$; so, since a is both positive and less than 2, we have $a < \sqrt{2a} < 2$.

(b) Prove that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \dots$$

converges.

Solution: The sequence can be defined recursively by

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n} \text{ for } n \geq 1.$$

Clearly all a_n are positive. We prove by induction on n that $a_n < 2$. The base case is clear. For the inductive step, since (by induction, and earlier observation about positivity) we have $0 < a_n < 2$, we have from part a) that $\sqrt{2a_n} < 2$, i.e. that $a_{n+1} < 2$, completing the induction.

We also observe that (a_n) is increasing. Indeed, since $0 < a_n < 2$ (as proved above), we have immediately from part a) that $a_n < \sqrt{2a_n} = a_{n+1}$.

So $(a_n)_{n=1}^{\infty}$ is increasing and bounded above, and so tends to a limit.

(c) Let a_n be the nth term of the above sequence, and let $\ell = \lim_{n\to\infty} a_n$. Carefully applying a theorem proved in lectures, find ℓ.

Solution: Consider the function $f(x) = \sqrt{2x}$ on $(0, \infty)$. It's continuous, and so in particular is continuous at ℓ (which is certainly positive). Now since $(a_n) \to \ell$, we have that $(f(a_n)) \to f(\ell) = \sqrt{2\ell}$. But also $(f(a_n)) = (a_{n+1}) \to \ell$, so by uniqueness of limits we have $\ell = \sqrt{2\ell}$, implying $\ell = 2$.

6. This question provides a useful estimate on $n!$: $n! \approx (n/e)^n$.

(a) Show that if $f : [1, \infty)$ is increasing then

$$f(1) + \cdots + f(n-1) < \int_1^n f(x)dx < f(2) + \cdots + f(n).$$

Solution: An upper bound on $\int_1^n f(x)dx$ is provided by $U(f, P)$ where P is the partition $\{1, 2, 3, \dots, n\}$. Since f is increasing, $\sup\{f(x) : x \in [i, i+1]\} = f(i+1)$, and so

$$U(f, P) = f(2) + \cdots + f(n)$$

(notice that the difference between consecutive points of the partition is 1). This shows

$$\int_1^n f(x)dx < f(2) + \cdots + f(n).$$

For the other direction, a lower bound on $\int_1^n f(x)dx$ is provided by $L(f, P)$ where P is the same partition as before. Again since f is increasing, $\inf\{f(x) : x \in [i, i+1]\} = f(i)$, and so

$$L(f, P) = f(1) + \cdots + f(n-1).$$

This shows

$$f(1) + \cdots + f(n-1) < \int_1^n f(x)dx.$$

(b) By taking $f = \log$ deduce that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

Solution: We have $\int \log x \; dx = x \log x - x$, so

$$\int_1^n \log x \; dx = [x \log x - x]_{x=1}^n = n \log n - n - 1 \log 1 + 1 = n \log n - n + 1.$$

We also have $\log 1 + \cdots + \log(n-1) = \log(n-1)!$ and $\log 2 + \cdots + \log n = \log n!$. Since log is increasing, we can apply the upper bound for part a) to get

$$n \log n - n + 1 \le \log n!$$

which on exponentiating yields

$$\frac{n^n}{e^{n-1}} < n!.$$

And applying the lower bound from part a) we have

$$\log(n-1)! < n\log n - n + 1$$

or, changing index from n to $n+1$,

$$\log n! < (n+1)\log(n+1) - n,$$

which on exponentiating yields

$$n! < \frac{(n+1)^{n+1}}{e^n}.$$

(c) Deduce that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

Solution: From the lower bound in part b) we have

$$\frac{\sqrt[n]{n!}}{n} > \left(\frac{1}{e}\right) e^1/n.$$

From the upper bound we have

$$\frac{\sqrt[n]{n!}}{n} < \left(\frac{1}{e}\right) \sqrt[n]{n+1}\left(1+\frac{1}{n}\right)$$

Since $e^{1/n} \to 1$ and $\sqrt[n]{n+1}\left(1+\frac{1}{n}\right) \to 1$ as $n \to \infty$, we get from the obvious squeeze theorem that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

7. The *Harmonic number* H_n is the number $H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. This exercise gives a very useful estimate on H_n, namely $H_n \approx \log n$.

(a) Notice that $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$ and

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}.$$

Generalize this: prove that for all $k \geq 0$, $H_{2^k} \geq 1 + \frac{k}{2}$ (and so $(H_n)_{n=1}^{\infty}$ diverges to $+\infty$).

Solution: We have

$$\begin{aligned} H_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k} \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots + \frac{1}{2^k}, \end{aligned}$$

where we lower bound 1/3 by 1/4 (so the sum has 2 1/4's), we lower bound each of 1/5, 1/6 and 1/7 by 1/8 (so the sum has 4 1/8's), and so on up to lower bounded each of $1/(2^{k-1}+1), 1/(2^{k-1}+2), \cdots 1/(2^{k-1}+(2^{k-1}-1))$ by $1/2^k$ (so the sum has 2^{k-1} $1/2^k$'s). So, continuing the chain of inequalities from above,

$$\begin{aligned} H_{2^k} &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots + \frac{1}{2^k} \\ &= 1 + \frac{1}{2} + 2\frac{1}{4} + 4\frac{1}{8} + \cdots + 2^{k-1}\frac{1}{2^k} \\ &= 1 + \frac{k}{2}, \end{aligned}$$

as claimed.

(b) Prove that for all natural numbers n,

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}.$$

Solution: We'll prove

$$\frac{1}{x+1} < \log(x+1) - \log x < \frac{1}{x}$$

for $x \in [1, \infty)$, which is equivalent to

$$\frac{1}{(1/y)+1} < \log(1/y+1) - \log 1/y < \frac{1}{1/y}$$

for $y \in (0, 1]$. After a little re-arranging this becomes

$$1 - \frac{1}{1+y} < \log(1+y) < y$$

for $y \in (0, 1]$.

To prove this, consider function $f(y) = 1 - 1/(1+y)$, $g(y) = \log(1+y)$ and $h(y) = y$ defined on $[0, 1]$. They all agree at 0 (with value 0), but on $(0, 1]$ we have $f'(y) = \frac{1}{(1+y)^2}$, $g'(y) = \frac{1}{1+y}$, $h'(y) = 1$, so $f'(y) < g'(y) < h'(y)$ on $(0, 1]$. from basic properties of the derivative it follows that $f(y) < h(y) < g(y)$ on $(0, 1]$, as required.

(c) Deduce from part (b) that the sequence $(H_n - \log n)_{n=2}^{\infty}$ is decreasing and bounded below by 0.

Solution: First we check that the sequence is decreasing: we have

$$H_{n+1} - \log(n+1) < H_n - \log n$$

if and only if

$$\frac{1}{n+1} < \log(n+1) - \log n,$$

which was established in part b).

To get that it is bounded below by 0, we use the other inequality from part b) to get, via a telescoping sum,

$$\begin{aligned} \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} &> (\log 2 - \log 1) + (\log 3 - \log 2) + \cdots + (\log(n+1) - \log n) \\ &= \log(n+1) \end{aligned}$$

so that

$$H_n - \log n > \log(n+1) - \log n > \frac{1}{n+1} > 0$$

(the penultimate inequality above using the first inequality from part b)).

(d) Explain why you can deduce that there is a number $\gamma \geq 0$ such that

$$\lim_{n\to\infty} (H_n - \log n) = \gamma.$$

(This number is known as the *Euler-Mascheroni constant*, and is approximately 0.57721. It is not known whether γ is rational or irrational.)

Solution: $(H_n - \log n)_{n=2}^{\infty}$ is decreasing and bounded below by 0, so by a theorem proved in the lectures we know that

$$\lim_{n\to\infty} (H_n - \log n) \text{ exists and equals } \inf\{H_n - \log n : n \in \mathbb{N}\}, \text{ which is } \geq 0.$$

(Actually, in the lectures we proved that if (a_n) is increasing and bounded above then $\lim_{n\to\infty} a_n$ exists and equals $\inf\{a_n : n \in \mathbb{N}\}$, but the proof of the analogous statement used here is identical.