Math 10860, Honors Calculus 2

Midterm 1 information

Spring 2019

The first midterm will be

Monday March 2, in class.

It will cover everything that we have covered in class this semester up to the beginning of class on Wednesday, February 26. In the course notes, that is Chapters 10, 11 and 12; in terms of homeworks and quizzes, it is everything that was covered on homeworks 1 through 6 and quizzes 1 through 6.

Some of the exam will be problems, some will be definitions, and some will be proofs of results from class. Here are the results that I am thinking of:

- Every lower Darboux sum is at most as large as every upper Darboux sum (and, closely related: refining a partition increases the lower Darboux sum and decreases the upper Darboux sum)
- The sum of integrable functions is integrable
- If f is integrable on $[a,b]$, and $c \in (a,b)$, then f is integrable on both $[a,c]$ and $[c,b]$, and (going the other direction), if f is integrable on $[a,c]$ and $[c,b]$ then f is integrable on $[a,b]$
- cf is integrable when f is
- Monotone functions are integrable
- Uniformly continuous functions are integrable[1]
- The comparison theorem for improper integrals
- $\int_a^x f$ is continuous as a function of x
- FTOC part 2 [2]
- The inverse of a continuous function is continuous[3]

[1]While you should understand the proof that uniform continuity on a closed interval implies continuity, I won't ask about the proof on the exam.

[2]While you should understand the proof of FTOC part 1, I won't ask about the proof on the exam.

[3]While you should understand the proof that the inverse of a differentiable function is differentiable (modulo some obvious condition), I won't ask about the proof on the exam.

- If a continuous function defined on an interval is invertible, it must be monotone
- All solutions to $f' = f$
- All solutions to $f'' + f = 0$

The rest of this document is a collection of practice problems. The first set are focussed on definitions, and you are supposed to find these very easy! The rest are problems that vary from reasonable to somewhat harder to just plain hard. There are no practice questions asking you to proof various facts from the notes — you should be reviewing these in the notes.

I will endeavor as I write the exam to have no more than one problem that is in the "just plain hard" category!

Definitions

1. (a) Say what is a *partition* of an interval $[a, b]$ with $a < b$.

   Solution: A partition of $[a, b]$ is a set of numbers $\{t_0, t_1, \ldots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$.

   (b) For a partition P of $[a, b]$, and a bounded function $f : [a, b] \to \mathbb{R}$, what is the *upper Darboux sum* $U(f, P)$?

   Solution: The upper Darboux sum $U(f, P)$ is

   $$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

   where $M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\}$.

   (c) If P is a partition of $[a, b]$ and Q is another partition of $[a, b]$ that includes all the points of P (and perhaps some more), what is the relationship between $L(f, P), U(f, P), L(f, Q)$ and $U(f, Q)$?

   Solution: We have the relationship

   $$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

   (d) Give the definition of bounded f being *integrable* on $[a, b]$, and say what the value of the integral is.

   Solution: f is integrable on $[a, b]$ if

   $$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} = \inf\{U(f, P) : P \text{ a partition of } [a, b]\},$$

   and the value of the integral is the common value above (either $\sup\{L(f, P) : P \text{ a partition of } [a, b]\}$ or $\inf\{U(f, P) : P \text{ a partition of } [a, b]\}$)

Note that is is *not* correct to *define* f being integrable by "for all $\varepsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$"; this is a *fact* about the integral, derived from the definition, but not the definition.

Nor is it correct to begin your answer "if f is integrable then ...". If you do, then you are giving a *consequence* of integrability, not the definition.

2. (a) State the first fundamental theorem of calculus, paying attention to the necessary hypotheses.

Solution: The first fundamental theorem of calculus states that

> If f is integrable on $[a,b]$, and $F : [a,b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f(t)dt$, then if f is continuous at c it follows that F is differentiable at c, and that $F'(c) = f(c)$.

(Or, slightly more generally:

> If f is integrable on an interval I, a is in I, and $F : I \to \mathbb{R}$ is defined by $F(x) = \int_a^x f(t)dt$, then if f is continuous at c it follows that F is differentiable at c, and that $F'(c) = f(c)$.)

Note that this is a stronger statement than "if f is continuous on $[a,b]$ then F is differentiable on $[a,b]$ and $F' = f$"; one of the strengths of the fundamental theorem of calculus is that to conclude differentiability of F at a point, one only needs to know continuity of f at a single point, not on any interval.

(b) State the second fundamental theorem of calculus, paying attention to the necessary hypotheses.

Solution: The second fundamental theorem of calculus states that

> If f is integrable on $[a,b]$, and g is a function $g : [a,b] \to \mathbb{R}$ such that $g' = f$ on $[a,b]$, then $\int_a^b f(x)dx = g(b) - g(a)$.

Note that one needs the hypothesis of integrability. Without it, the theorem would imply (among other things) that any function f that has an antiderivative is automatically integrable. But this is not the case; look up "Volterra's function" for an example.

3. Let $f : [a,b] \to \mathbb{R}$ be bounded and integrable. Set $F(x) = \int_a^x f(t)\ dt$ for $x \in [a,b]$. For each of the following, either *briefly* justify an affirmative answer, or give either an example or a *brief* explanation to show that the answer is negative.

(a) F is integrable on $[a,b]$.

Solution: Yes. We know that for any integrable f, the function F defined as above as the integral of f is continuous; and we also know that any continuous function is integrable.

(b) F is differentiable on $[a, b]$, with $F' = f$.

Solution: No. Certainly, if we assume that f is continuous, then F is differentiable, with $F' = f$ (by the fundamental theorem of calculus); but if f is not continuous, then there's no guarantee that F is differentiable or, if it is, that $F' = f$.

For example, if $f = 0$ except at one point x, where it takes value 1, then $F = 0$ everywhere, so is differentiable, but $F' \neq f$ at x.

Or, if $f = 0$ on $x \leq 0$ and $f = 1$ on $x > 0$, then F is easily seen not to be differentiable at 0.

(c) F is increasing on $[a, b]$

Solution: No. If f is positive, then F will be increasing (it is accumulating area), but if f is ever negative it will lead to decreasing F. For example, consider $f = -1$ on $[0, 1]$; the associated F is decreasing.

(d) Knowing F, it is possible to uniquely determine f

Solution: No. If f is known to be continuous then it is uniquely determined by F, since f can be recovered from F by $F' = f$ (the derivative is unique).

But if f is not known to be continuous, then F does not determine f, the reason being that if $f^\star$ is obtained from f by changing the function at finitely many values, then the integral of $f^\star$ is equal to the integral of f over any interval. So F cannot distinguish between f and $f^\star$.

4. (a) What does it mean to say that a function f is *one-to-one* on its domain?

Solution: This means that for every $b \in \text{Domain}(f)$ there exists a *unique* a in $\text{Range}(f)$ such that $f(a) = b$. An equivalent definition is that if a_1, a_2 in the domain of f are distinct (satisfy $a_1 \neq a_2$), then $f(a_1) \neq f(a_2)$.

(b) What can you say about a continuous, one-to-one function whose domain is an interval?

Solution: One can say many things (such that it is invertible, and that the inverse is continuous), but what I was thinking of was the substantial fact that we proved, that such an f must be monotone (either strictly increasing or strictly decreasing).

(c) Let $f : (a, b) \to \mathbb{R}$ be continuous and invertible with inverse f^{-1}. Let c be a point in the domain of f^{-1}. Is f^{-1} necessarily continuous at c, and if not, what extra condition(s) need to be added to ensure that it is?

Solution: Yes, f^{-1} is necessarily continuous at c; we proved this in class.

(d) Let $f : (a, b) \to \mathbb{R}$ be differentiable and invertible with inverse f^{-1}. Let c be a point in the domain of f^{-1}. Is f^{-1} necessarily differentiable at c, and if not, what extra condition(s) need to be added to ensure that it is?

Solution: No, f^{-1} is not necessarily differentiable at c. If the condition that $f'(f^{-1}(c)) \neq 0$ is added, then (as we proved in class) it becomes the case that f^{-1} is differentiable at c.

Note that it is not a proper answer here to say:

> "If $f'(f^{-1}(c)) = 0$ then f^{-1} is not differentiable at c."

and leave it at that; this leaves open the possibility that there are *also* some places where $f'(f^{-1}(c)) \neq 0$ for which f^{-1} is not differentiable at c.

5. (a) Give the definition of the function log, including a statement of its domain and range.

   Solution: $\log : (0, \infty) \to \mathbb{R}$ is defined by

$$\log x = \int_1^x \frac{dt}{t}.$$

   Its domain is $(0, \infty)$ and its range is $\mathbb{R}$.

   (b) Give the definition of the function exp, including a statement of its domain and range.

   Solution: $\exp : \mathbb{R} \to (0, \infty)$ is defined by

$$\exp x = \log^{-1} x$$

   (exp is the inverse of log). Its domain is $\mathbb{R}$ and its range is $(0, \infty)$.

   (c) How is the number e defined?

   Solution: e can either be defined to be the unique number x such that $\log x = 1$ (i.e., it is defined by the relation $\int_1^e \frac{dt}{t} = 1$), or, equivalently, it can be defined as $\exp(1)$.

   (d) For a positive number a and a real number b, what does the expression a^b mean?

   Solution: It means $\exp(b \log a)$ (or $e^{b \log a}$)

6. (a) How is the number π defined?

   Solution: Either π is defined by the relation

$$\frac{\pi}{2} = \int_{-1}^{1} \sqrt{1 - t^2} dt,$$

   or by $\pi = 2 \int_{-1}^{1} \sqrt{1 - t^2} dt$.

(b) How is the function cos defined? (This needs to be a multi-step definition!)

Solution: On $[0, \pi]$, $\cos x$ is defined to be $A^{-1}(x/2)$ where $A : [-1, 1] \to [0, \pi/2]$ is defined by

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-x^2}\, dx.$$

For $x \in [\pi, 2\pi]$ we defined $\cos x$ to be equal to $\cos(2\pi - x)$.

For $x \notin [0, 2\pi]$ we define $\cos x$ to be $\cos(x + 2n\pi)$ where n is any integer satisfying $x + 2n\pi \in [0, 2\pi]$.

7. Complete these identities.

(a) $e^{a+b} = \ldots$

Solution: $e^{a+b} = e^a e^b$.

(b) $\log a^b = \ldots$

Solution: $\log a^b = b \log a$.

(c) $\sin(x - y) = \ldots$

Solution: $\sin(x - y) = \sin x \cos y - \sin y \cos x$.

8. (a) Give the definition of a function f being continuous at a point a.

Solution: f is continuous at a if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all x, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

(b) Give the definition of a function f being continuous on an interval I.

Solution: f is continuous on an interval I if it is continuous at every point in I.

(c) Give the definition of a function f being *uniformly* continuous on an interval I.

Solution: f is uniformly continuous on an interval I if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all x, y in the interval, if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

(d) What is the difference between the last two definitions?

Solution: For continuity on an interval, for each x in the interval, for each $\varepsilon > 0$ we have to find a $\delta > 0$ with the property that for all y within δ of x, $f(y)$ is within ε of $f(x)$; δ is allowed to depend on x as well as on ε.

For uniform continuity on an interval, for each $\varepsilon > 0$ we have to find a $\delta > 0$ with the property that for all x, y within δ of each other, $f(y)$ is within ε of $f(x)$; δ is only allowed to depend ε, not on x.

Problems

1. (a) Express $\sin x - \sin y$ as a product of two trigonometric functions.

   Solution: We have

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

   and

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

   so

$$\sin(a+b) - \sin(a-b) = 2\cos a \sin b.$$

   Setting $x = a+b$ and $y = a-b$, get $a = (x+y)/2$ and $b = (x-y)/2$, so

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

   (b) Express $\sin\left(k+\frac{1}{2}\right)x - \sin\left(k-\frac{1}{2}\right)x$ as a product of two trigonometric functions.

   Solution: Apply the previous result with the role of x played by $(k+1/2)x$ and the role of y played by $(k-1/2)x$. We get

$$\sin\left(k+\frac{1}{2}\right)x - \sin\left(k-\frac{1}{2}\right)x = 2\cos kx \sin\frac{x}{2}.$$

   (Or, easier: write $\sin\left(k+\frac{1}{2}\right)x - \sin\left(k-\frac{1}{2}\right)x$ as $\sin\left(kx+\frac{x}{2}\right) - \sin\left(kx-\frac{x}{2}\right)$, and use the angle summation formulae. One gets the same result with this method.)

   (c) Prove that for natural numbers n,

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} - \frac{1}{2}.$$

   Solution: A proof by induction is possible, with part (b) in the case $k=1$ giving the base case, and part (b) in general giving the induction step.

   Or, easier: we have, using the result of the last part for the first line, and the fact that the sum is telescoping for the second,

$$\begin{aligned} 2\cos x \sin\frac{x}{2} + \cdots + 2\cos nx \sin\frac{x}{2} &= \sum_{k=1}^{n}\left(\sin\left(k+\frac{1}{2}\right)x - \sin\left(k-\frac{1}{2}\right)x\right) \\ &= \sin\left(n+\frac{1}{2}\right)x - \sin\frac{x}{2}. \end{aligned}$$

   Dividing both sides by $2\sin\frac{x}{2}$ gives the claimed identity.

2. Recall that we defined sin first on the interval $[0,\pi]$ (via some complicated expression), and we observed that $\sin 0 = \sin\pi = 0$. We then extended the definition of sin to

$[0, 2\pi]$ by declaring that for all $\theta \in [\pi, 2\pi]$, $\sin\theta = -\sin(2\pi - \theta)$[4]. Finally, we defined sin for all reals by periodic extension: given any $\theta \in \mathbb{R}$, there is an integer n such that $\theta + 2\pi n \in [0, 2\pi]$; declare $\sin\theta$ to be the same value as $\sin(2n\pi + \theta)$.[5]

More generally, let $f : [0, \pi] \to \mathbb{R}$ be any function (continuous or otherwise) satisfying $f(0) = f(\pi) = 0$. Extend f to $[0, 2\pi]$ by declaring that for all $x \in [\pi, 2\pi]$, $f(x) = -f(2\pi - x)$; then extend the definition of f to all reals by periodic extension (as described above)[6]. Prove that f is an odd function.[7]

Solution: Consider an arbitrary $x \in \mathbb{R}$. There is an integer n such that $x+2n\pi \in [0, 2\pi]$. Consider two cases.

Case 1 ($x + 2n\pi \in [0, \pi]$). In this case by definition we have

$$f(2\pi - (x + 2n\pi)) = -f(x + 2n\pi).$$

But by periodicity, $f(2\pi - (x + 2n\pi)) = f(-x + 2(1 - n)\pi) = f(-x)$ and $f(x + 2n\pi) = f(x)$, so

$$f(-x) = -f(x),$$

as required.

Case 2 ($x + 2n\pi \in [\pi, 2\pi]$). In this case by definition we have

$$f(x + 2n\pi) = -f(2\pi - (x + 2n\pi)).$$

But now exactly as in Case 1 we have $f(-x) = -f(x)$.

3. Suppose that f is integrable on $[a, b]$, with $a < b$.

   (a) Show that there is $x \in [a, b]$ with

$$\int_a^x f = \int_x^b f.$$

Solution: Consider the function $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f$. We know from a theorem from class that F is continuous (**NOTE**: this is **not** the fundamental theorem of calculus!). Since $F(0) = 0$ and $F(b) = \int_a^b f$, from the intermediate value theorem we have that there is an $x \in [a, b]$ with $F(x) = \frac{1}{2}\int_a^x f$. Basic linearity properties of the integral now show that for any such x,

$$\int_a^x f = \int_x^b f.$$

[4] Note that we have to be careful at $\theta = \pi$, where sin has already been defined to be 0; but $2\pi - \pi = \pi$, so at $\theta = \pi$ the new definition says that $\sin\pi = -\sin\pi = -0 = 0$, which agrees with the old definition

[5] If θ is not an integer multiple of 2π, there is a unique n that works. If θ is a multiple of 2π then there two such n's, say n_1 and n_2, with $2\pi n_1 = 0$ and $2\pi n_2 = 2\pi$. This does not create an ambiguity since $\sin 0 = \sin 2\pi = 0$.

[6] As with sin, all this is well-defined, and for the same reasons exactly.

[7] This justifies our claim in class that $\sin(-x) = -\sin(x)$, that was justifiably questioned at the time it was made. One can similarly argue that cos is an even function.

(To be a little more precise: if $\int_a^b f \neq 0$ then the intermediate value theorem gives that there is $x \in (a,b)$ with $\int_a^x f = \int_x^b f$; and if $\int_a^b f = 0$ then we may take either $x = a$ or $x = b$ to get $\int_a^x f = \int_x^b f$).

(b) What condition on f ensures that it is possible to find such an x in (a,b)? Give examples to show that if this condition is not satisfied, then

- it is sometimes possible to find such an x in (a,b), and
- it is sometimes *not* possible to find such an x in (a,b).

Solution: As observed in part (a), if $\int_a^b f \neq 0$ then yes, such x can be guaranteed to be in (a,b). If $\int_a^b f = 0$ then no such guarantee can be made. Consider, for example, f given by $f(x) = \sin x$ with $a = 0$ and $b = 2\pi$. We have $\int_0^{2\pi} \sin x dx = 0$, and both $x = 0$ and $x = 2\pi$ have the property that

$$\int_0^x \sin t dt = \int_x^{2\pi} \sin t dt.$$

But if $x \in (0, 2\pi)$ we have

$$\int_0^x \sin t dt = 1 - \cos x \quad \text{and} \quad \int_x^{2\pi} \sin t dt = \cos x - 1$$

so $\int_0^x \sin t dt = \int_x^{2\pi} \sin t dt$ is equivalent to $1 - \cos x = \cos x - 1$ or $\cos x = 1$, which has no solution in $(0, 2\pi)$.

On the other hand, the constant zero function is an example of a function with $\int_a^b f = 0$, but for which *every* x satisfies the condition of the question.

4. Let $f : [a,b] \to \mathbb{R}$ be monotone.

(a) Explain why f is bounded.

Solution: If f is monotone increasing then (by definition) $f(a) < f(x) < f(b)$ for all $x \in (a,b)$, so f is bounded above by $f(b)$, and below by $f(a)$. Similarly if f is monotone decreasing then it is bounded above by $f(a)$, and below by $f(b)$.

(b) Prove that $\int_a^b f$ exists.

Solution: This was (basically) Homework 1, Question 5 (to which a solution has already been posted).

5. (a) Let $f : [a,b] \to \mathbb{R}$ be a differentiable function that is never 0. Assuming[8] that f'/f

[8] There is a subtlety in this question. There's a (hopefully fairly obvious) function g with $g' = f'/f$, and once g has been identified, the integral can be evaluated using FTOC part 2. The subtlety is that even after a g with $g' = f'/f$ has been found, there is no guarantee that f'/f is integrable. There are examples of differentiable functions whose derivatives are not integrable. If V is such a function (I use V here because the first example of such a function was discovered by Volterra, and is called *Volterra's function*) then while it is true to say that $V = \int V'$ (V is a primitive of V'), it is *not* true to say that $\int_a^b V' = V(b) - V(a)$, since the left-hand side exists but the right-hand side doesn't. Remember that the FTOC part 2 says that if there's a function g with $g' = f$ *and* f is integrable then $\int_a^b f = g(b) - g(a)$. So it is necessary to add the caveat "assuming that f'/f is integrable on $[a,b]$" (which I had left out in an earlier version of this document).

is integrable on $[a, b]$, find a very simple expression for

$$\int_a^b \frac{f'(t)}{f(t)} dt.$$

Solution: Note that f is continuous (since it is differentiable) and never zero, so by IVT it is either always positive or always negative. If it is always positive then $\log f(x)$ exists and is continuous, and

$$(\log f(x))' = \frac{f'(x)}{f(x)}.$$

If it is always negative then $\log -f(x)$ exists and is continuous, and

$$(\log -f(x))' = \frac{-f'(x)}{-f(x)} = \frac{f'(x)}{f(x)}.$$

So in either case

$$(\log |f(x)|)' = \frac{f'(x)}{f(x)}.$$

BY FTOC(2),

$$\int_a^b \frac{f'(t)}{f(t)} = \log |f(x)|.$$

(b) *Use part (a)* to evaluate

$$\int_{\pi/6}^{\pi/3} \frac{dx}{(\sin x)(\cos x)}.$$

Solution: $1/(\sin x \cos x) = \sec^2 x / \tan x$, and so since $\tan' = \sec^2$ and $\tan$ is never 0 on $[\pi/6, \pi/3]$ (in fact is always positive) we get

$$\int_{\pi/6}^{\pi/3} \frac{dx}{(\sin x)(\cos x)} = \log \tan(\pi/3) - \log \tan(\pi/6) = \log \sqrt{3} - \log(\sqrt{3}/3) = \log 3.$$

6. Suppose that f is a differentiable function, with $f(0) = 0$ and $0 < f' \leq 1$. Prove that for all $x \geq 0$ we have

$$\int_0^x f^3 \leq \left(\int_0^x f \right)^2$$

(a) assuming that f' is integrable, and

(b) not assuming that f' is integrable (it may not be — there exist examples of functions whose derivatives are not integrable, such as Volterra's function (which has a wikipedia page)).

Solution: Probably the most sensible approach does not make any assumptions on f', so it answers parts (b) and (a) simulytaneously.

We will use a lemma that we saw last semester, as an application of the Mean Value Theorem:

If $h_1, h_2 : [a, \infty) \to \mathbb{R}$ are both differentiable, if $h_1(a) = h_2(a)$, and if $h_1'(x) \geq h_2'(x)$ on $[a, \infty)$ (considering derivative from above at a) then $h_1(x) \geq h_2(x)$ on $[a, \infty)$.

(The proof is: suppose not, so that there is some $b > a$ with $h_1(b) < h_2(b)$. Apply the MVT to the function $h_1 - h_2$ on $[a, b]$ to find a $c \in (a, b)$ with

$$(h_1 - h_2)'(c) = \frac{(h_1 - h_2)(b) - (h_1 - h_2)(a)}{b - a} < 0,$$

so $h_1'(c) < h_2'(c)$, a contradiction.)

Note that since f is differentiable, it is continuous, so f^3 is continuous, and so the function h_2 defined by $h_2(x) = \int_0^x f^3$ is differentiable, and the function h_1 defined by $h_1(x) = \left(\int_0^x f\right)^2$ is also differentiable (being the composition of the square function, which is differentiable, and the function defined by $\int_0^x f$, which is differentiable by the continuity of f).

We have $h_2(0) = h_1(0) = 0$, so to establish $h_2(x) \leq h_1(x)$ for $x \geq 0$, by the lemma it suffices to show $h_2'(x) \leq h_1'(x)$ for $x \geq 0$. By the fundamental theorem of calculus (and the chain rule) this is equivalent o

$$f(x)^3 \leq 2\left(\int_0^x f\right) f(x).$$

We will now establish this. Since $f(0) = 0$ and $f' > 0$, it follows that $f(x) > 0$ for $x > 0$ (an easy application of the mean value theorem). So proving the above inequality (which is easily seen to be an equality at $x = 0$) is equivalent to proving

$$f(x)^2 \leq 2\left(\int_0^x f\right)$$

for $x > 0$, which is implied by

$$f(x)^2 \leq 2\left(\int_0^x f\right)$$

for $x \geq 0$. So this is what we now work to establish. We apply the lemma above with now the role of h_2 being played by $f(x)^2$ (which, as we have already noted, is differentiable) and the role of h_2 being played by $2\left(\int_0^x f\right)$ (also differentiable). We have $h_2(0) = h_1(0) = 0$, so to establish $h_2(x) \leq h_1(x)$ for $x \geq 0$, by the lemma it suffices to show $h_2'(x) \leq h_1'(x)$ for $x \geq 0$, which is equivalent to

$$2f(x)f'(x) \leq 2f(x).$$

This is evident at $x = 0$; for $x > 0$ it is equivalent (via $f(x) > 0$ for $x > 0$) to $f'(x) \leq 1$, which is given.

We conclude that indeed $\int_0^x f^3 \leq \left(\int_0^x f\right)^2$.

7. (a) Suppose that f and g are both one-to-one. Show that $f \circ g$ is also one-to-one.

Solution: Let a, b be two points in the domain of g, such that $g(a), g(b)$ are in the domain of f (i.e., a, b are two points in the domain of $f \circ g$). By one-to-oneness of g we have $g(a) \neq g(b)$. But then by one-to-oneness of f we have $f(g(a)) \neq f(g(b))$. So, $a \neq b$ implies $(f \circ g)(a) \neq (f \circ g)(b)$, that is, $f \circ g$ is one-to-one.

(b) Suppose that f and g are both one-to-one. Explicitly express $(f \circ g)^{-1}$ in terms of f^{-1} and g^{-1}.

Solution: Suppose c is in the domain of $(f \circ g)^{-1}$. That means that c is in the range of $f \circ g$. This means that there is (unique, by one-to-oneness of f) b such that $(b, c) \in f$ with also $(a, b) \in g$. The function that sends c to a is $g^{-1} \circ f^{-1}$ (first send c to b, then send b to a). So

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

(c) If $g(x) = 1 + f(x)$, explicitly express g^{-1} in terms of f^{-1}.

Solution: Here g is $p \circ f$ where $p(x) = 1 + x$. We have $p^{-1}(x) = x - 1$, so (by part (b))

$$g^{-1}(x) = (p \circ f)^{-1}(x) = (f^{-1} \circ p^{-1})(x) = f^{-1}(x - 1).$$

8. (a) Decide which of these integrals exist (have finite values): $\int_0^1 \frac{dx}{x\sqrt{1+x}}$, $\int_1^\infty \frac{dx}{x\sqrt{1+x}}$, $\int_0^\infty \frac{dx}{x\sqrt{1+x}}$.

Solution: For $0 < x \leq 1$ we have

$$\frac{1}{2x} \leq \frac{1}{x\sqrt{1+x}}$$

(after a little algebra this is seen to actually hold for $0 < x \leq 3$). We also have

$$\int_\varepsilon^1 \frac{dx}{2x} = \frac{1}{2}\left[\log x\right]_{x=\varepsilon}^1 = -\log \varepsilon = \log \frac{1}{\varepsilon}.$$

It follows that

$$\int_\varepsilon^1 \frac{dx}{x\sqrt{1+x}} \geq \log \frac{1}{\varepsilon}.$$

Since $\log(1/\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$, it follows that the improper integral

$$\int_0^1 \frac{dx}{x\sqrt{1+x}}$$

does not exist.

On the other hand we have

$$\frac{1}{x^{5/4}} \geq \frac{1}{x\sqrt{1+x}} \geq 0$$

for all $x \geq 1$, so by comparison with $\int_1^\infty \frac{dx}{x^{5/4}}$ (which is finite, as we have seen in class) we get that

$$\int_1^\infty \frac{dx}{x\sqrt{1+x}}$$

exists.

For

$$\int_0^\infty \frac{dx}{x\sqrt{1+x}}$$

to exist we need both $\int_0^1 \frac{dx}{x\sqrt{1+x}}$ and $\int_1^\infty \frac{dx}{x\sqrt{1+x}}$ to exist; since only one of them does, the final integral does not exist.

(b) Find the value of

$$\int_0^a \frac{dx}{x^r}$$

for all $r \in (0,1)$ and $a > 0$ (your answer will depend on both a and r).

Solution:

$$\int_\varepsilon^a \frac{dx}{x^r} = \left[\frac{x^{1-r}}{1-r}\right]_{x=\varepsilon}^a = \frac{a^{1-r} - \varepsilon^{1-r}}{1-r}.$$

Since $\varepsilon^{1-r} \to 0$ as $\varepsilon \to 0$ for all $r \in (0,1)$, we get that

$$\int_0^a \frac{dx}{x^r} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^a \frac{dx}{x^r} = \frac{a^{1-r}}{1-r}.$$