Math 10860, Honors Calculus 2

Midterm 2 practice problem solutions

Spring 2020

1. Let

$$I_n = \int x^n \sin x \; dx.$$

(a) Find I_0 and I_1.
Solution: $I_0 = \int \sin x \; dx = -cosx$, while (via integration by parts)

$$I_1 = \int x \sin x \; dx = -x \cos x + \int \cos x \; dx = -x \cos x + \sin x.$$

(b) Find a reduction formula that expresses I_{n+2} in terms of I_n for $n \geq 0$.
Solution: We have, doing integration by parts twice,

$$\begin{aligned} I_{n+2} &= \int x^{n+2} \sin x \; dx \\ &= -x^{n+2} \cos x + (n+2) \int x^{n+1} \cos x \; dx \\ &= -x^{n+2} \cos x + (n+2) \left[x^{n+1} \sin x - (n+1) \int x^n \sin x \; dx \right] \\ &= -x^{n+2} \cos x + (n+2) x^{n+1} \sin x - (n+2)(n+1) I_n \end{aligned}$$

(c) Find $\int x^5 \sin x \; dx$.
Solution:

$$\begin{aligned} \int x^5 \sin x \; dx &= I_5 \\ &= -x^5 \cos x + 5x^4 \sin x - 20 I_3. \end{aligned}$$

Since

$$\begin{aligned} I_3 &= -x^3 \cos x + 3x^2 \sin x - 6 I_1 \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x \end{aligned}$$

we get

$$\int x^5 \sin x \; dx = -x^5 \cos x + 5x^4 \sin x + 20x^3 \cos x - 60x^2 \sin x - 120x \cos x + 120 \sin x.$$

2. Find the following integrals:

(a)

$$\int \log^3 x \; dx.$$

Solution: One approach is parts, with $dv = dx$ and $u = \log^3 x$, so $v = x$ and $du = (3\log^2 x)/x \; dx$, leading to

$$\int \log^3 x \; dx = x \log^3 x - 3 \int \log^2 x \; dx.$$

Now again we use parts, with $dv = dx$ and $u = \log^2 x$, so $v = x$ and $du = (2\log x)/x \; dx$, leading to

$$\int \log^3 x \; dx = x \log^3 x - 3x \log^2 x + 6 \int \log x \; dx.$$

Finally, use parts again, with $dv = dx$ and $u = \log x$, so $v = x$ and $du = 1/x \; dx$, leading to

$$\int \log^3 x \; dx = x \log^3 x - 3x \log^2 x + 6x \log x - 6x.$$

(b)

$$\int \frac{\sqrt{1-x}}{1-\sqrt{x}} \; dx.$$

Solution: Start with $u = \sqrt{x}$, so $du = (1/2\sqrt{x})dx = (1/2u)dx$, so $dx = 2u \; du$. Also $1 - x = 1 - u^2$, so integral becomes

$$2 \int \frac{u\sqrt{1-u^2}}{1-u} \; du.$$

Now try $u = \sin t$, so $du = \cos t \; dt$, to get

$$2 \int \frac{\sin t \cos^2 t}{1 - \sin t} \; dt = 2 \int \frac{\sin t (1 - \sin t)(1 + \sin t)}{1 - \sin t} \; dt = 2 \int \left(\sin t + \sin^2 t\right) \; dt$$

The value of this last is $-2\cos t + t - \frac{\sin 2t}{2} = -2\cos t + t - \sin t \cos t$. Going back to u, get

$$-2\cos(\sin^{-1} u) + \sin^{-1} u - u\cos(\sin^{-1} u) = -2\sqrt{1-u^2} + \sin^{-1} u - u\sqrt{1-u^2}.$$

Going back to x, get

$$-2\sqrt{1-x} + \sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}.$$

(Notice that there is a slight mis-match of domains here. The domain of the integrand is $[0, 1)$, while the domain of the function we obtained as the integral is $[0, 1]$.)

(c)

$$\int \frac{dx}{2+\tan x}.$$

Solution: Via the substitution $u = \tan x$ (so $du = \sec^2 x\ dx$, $dx = du/\sec^2 x = du/(1+\tan^2 x) = du/(1+u^2)$), we get

$$\begin{aligned}
\int \frac{dx}{2+\tan x} &= \int \frac{du}{(1+u^2)(2+u)} \\
&= \int \frac{A+Bu}{1+u^2}du + \int \frac{C}{2+u}du \quad \text{(form of partial fractions decomposition)} \\
&= \int \frac{2-u}{5(1+u^2)}du + \int \frac{1}{5(2+u)}du \quad \text{(this after some algebra)} \\
&= \int \frac{2}{5(1+u^2)}du - \int \frac{u}{5(1+u^2)}du + \int \frac{1}{5(2+u)} \\
&= \frac{2}{5}\arctan u - \frac{1}{10}\log(1+u^2) + \frac{1}{5}\log|2+u| \\
&= \frac{2x}{5} - \frac{1}{5}\log\sec x + \frac{1}{5}\log|2+\tan x|.
\end{aligned}$$

We could stand to be a little more careful here. When we make a substitution, it should be invertible. But $u = \tan x$ isn't invertible. So technically, we need to consider the problem separately on each of the domains

$$\cdots, (-3pi/2, -\pi/2), (-\pi/2, \pi/2), (\pi/2, 3\pi/2), \cdots$$

(on each of which, tan is invertible). On each such domain, everything goes exactly as before, until we come to simplify $\arctan u = \arctan\tan x$. We said this equal x; that is only true on the domain $(-\pi/2, \pi/2)$. On, for example, the domain $(3\pi/2, 5\pi/2)$, we have

$$\arctan\tan x = x - 2\pi$$

(arctan returns a value between $-\pi/2$ and $\pi/2$, and $x - 2\pi$ is exactly the value in the interval $(-\pi/2, \pi/2)$ that has the same tan value as x). So on the domain $(3\pi/2, 5\pi/2)$ we get a different antiderivative. But it differs from the one we got on $(-\pi/2, \pi/2)$ by a constant, namely $-4\pi/5$, so it really doesn't differ.

The same thing happens on all the open intervals that make up the domain of tan; so the antiderivative we found, works everywhere.

(d)

$$\int \frac{x^6+x^5-2x^4-x^3+8x^2-4x+5}{(x-1)^2(x+1)^3}\ dx.$$

Solution: Start with polynomial long division to get

$$\frac{x^6+x^5-2x^4-x^3+8x^2-4x+5}{(x-1)^2(x+1)^3} = x + \frac{x^3+7x^2-5x+5}{(x-1)^2(x+1)^3}.$$

The form of the partial fractions decomposition of $(x^3+7x^2-5x+5)/((x-1)^2(x+1)^3)$ is

$$\frac{x^3+7x^2-5x+5}{(x-1)^2(x+1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}.$$

After multiplying both sides above by $(x-1)^2(x+1)^3$, expanding out both sides, equating coefficients of same powers of x to get 5 equations in 5 unknowns, and solving, one gets $A = C = D = 0$, $B = 1$, $E = 4$, i.e.

$$\frac{x^6+x^5-2x^4-x^3+8x^2-4x+5}{(x-1)^2(x+1)^3} = x + \frac{1}{(x-1)^2} + \frac{4}{x+1)^3}.$$

So

$$\int \frac{x^6+x^5-2x^4-x^3+8x^2-4x+5}{(x-1)^2(x+1)^3}dx = \frac{x^2}{2} - \frac{1}{x-1} - \frac{2}{(x+1)^2}.$$

(e)

$$\int_0^\pi (f(x)+f''(x))\sin x \; dx.$$

(Here f is defined and twice differentiable on $[0,\pi]$, with f'' continuous. Your answer will depend on f, of course.)

Solution: Split into two integrals:

$$\int_0^\pi (f(x)+f''(x))\sin x \; dx = \int_0^\pi f(x)\sin x \; dx + \int_0^\pi f''(x)\sin x \; dx.$$

Use integration by parts. For the first integral, use $u = f$ so $du = f'(x)dx$ and $dv = \sin x dx$ so $v = -\cos x$; for the second integral, use $u = \sin x$ so $du = \cos x dx$ and $dv = f''(x)dx$ so $v = f'(x)$ (here using continuity, so integrability, of f''). We get

$$\int_0^\pi f(x)\sin x \; dx = [-f(x)\cos x]_{x=0}^\pi + \int_0^\pi f'(x)\cos x \; dx = f(\pi)+f(0)+\int_0^\pi f'(x)\cos x dx$$

and

$$\int_0^\pi f''(x)\sin x \; dx = [f'(x)\sin x]_{x=0}^\pi - \int_0^\pi f'(x)\cos x \; dx.$$

Adding the two, the problem integral $\int_0^\pi f'(x)\cos x \; dx$ disappears, leaving

$$\int_0^\pi (f(x)+f''(x))\sin x \; dx = f(\pi)+f(0).$$

3. Recall that $\sinh x = \frac{e^x-e^{-x}}{2}$.

(a) Find the degree $2n+1$ Taylor polynomial of sinh about 0.

Solution: The kth derivative of $(e^x)/2$ at 0 is $1/2$, and the kth derivative of $(e^{-x})/2$ at 0 is $(-1)^k/2$. So the kth derivative of $\sinh x$ at 0 is

- 1 if k is odd,
- 0 if k is even.

We get

$$P_{2n+1,0,\sinh}(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}.$$

(b) Write down the Lagrange form of the remainder term $R_{2n+1,0,\sinh}(x)$.

Solution: Directly from Taylor's theorem,

$$R_{2n+1,0,\sinh}(x) = \frac{\sinh^{(2n+2)}(c)x^{2n+2}}{(2n+2)!} = \frac{(e^c - e^{-c})x^{2n+2}}{2(2n+2)!}$$

for some number c between 0 and x.

(c) Show that for all real x the remainder term $R_{2n+1,0,\sinh}(x)$ tends to 0 as n tends to infinity.

Solution: Fix a real number x. The function $\sinh x$ is increasing on its domain (all reals), so

$$\frac{(e^c - e^{-c})}{2} \leq \max\left\{\frac{(e^x - e^{-x})}{2}, \frac{(e^0 - e^{-0})}{2} \;(= 0)\right\}.$$

So

$$\left|\frac{(e^c - e^{-c})}{2}\right| \leq \left|\frac{(e^x - e^{-x})}{2}\right|.$$

Whatever this is, it is just some constant C_x (depending on x). Thus we have

$$|R_{2n+1,0,\sinh}(x)| \leq C_x \frac{|x|^{2n+2}}{(2n+2)!}.$$

We have proven that for all $x > 0$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, so

$$|R_{2n+1,0,\sinh}(x)| \to 0$$

as $n \to \infty$, as required.

(d) Write down a sum (using summation notation), with all the summands being rational numbers, whose value is within 10^{-10} of $\sinh 5$.

Solution: From part (c) we have

$$|R_{2n+1,0,\sinh}(5)| \leq \left|\frac{(e^5 - e^{-5})}{2}\right| \frac{5^{2n+2}}{(2n+2)!}$$

Using $e \leq 3$ we can write

$$|R_{2n+1,0,\sinh}(5)| \leq \frac{3^5 5^2}{2} \frac{25^n}{(2n+2)!}.$$

This first drops below 10^{-10} at $n = 15$, so the Taylor polynomial $P_{31,0,\sinh}(x)$ at $x = 5$ gives a number that is within 10^{-10} of $\sinh 5$. That is:

$$\sum_{n=0}^{15} \frac{5^{2n+1}}{(2n+1)!} = (\sinh 5) \pm 10^{-10}.$$

4. Suppose that a_i is the coefficient of $(x-a)^i$ in the Taylor polynomial of $f(x)$ at a (so $a_i = \frac{f^{(i)}(a)}{i!}$) and that b_i is the coefficient of $(x-a)^i$ in the Taylor polynomial of $g(x)$ at a. In terms of the a_i's and the b_i's, express the coefficient of $(x-a)^n$ in the Taylor polynomial of each of the following functions at a:

   (a) $2f - 3g$
   Solution: By linearity of the derivative, the nth derivative of $2f-3g$ at a is

$$2f^{(n)}(a) - 3g^{(n)}(a),$$

   so the coefficient of $(x-a)^n$ in the Taylor polynomial of $2f-3g$ at a is

$$\frac{2f^{(n)}(a) - 3g^{(n)}(a)}{n!} = 2a_n - 3b_n.$$

   (b) fg.
   Solution: We proved last semester that

$$(fg)^{(n)}(a) = \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(a) g^{(n-i)}(a)$$

   so that

$$\frac{(fg)^{(n)}(a)}{n!} = \sum_{i=0}^{n} \frac{\binom{n}{i} f^{(i)}(a) g^{(n-i)}(a)}{n!}$$

$$\sum_{i=0}^{n} \frac{\frac{n!}{i!(n-i)!} f^{(i)}(a) g^{(n-i)}(a)}{n!}$$

$$= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} \frac{g^{(n-i)}(a)}{(n-i)!}$$

$$= \sum_{i=0}^{n} a_i b_{n-i}.$$

   This is the coefficient of $(x-a)^n$ in the Taylor polynomial of fg at a.

   (c) $h(x) = \int_a^x f(t)\, dt$.
   Solution: For $n = 0$, have $h(a) = 0$. For $n > 0$,

$$h^{(n)}(x) = f^{(n-1)}(x)$$

   so

$$\frac{h^{(n)}(a)}{n!} = \frac{f^{(n-1)}(a)}{n!} = \frac{a_{n-1}}{n}.$$

   This is the coefficient of $(x-a)^n$ in the Taylor polynomial of h at a.

5. Compute the following sequence limits:

(a)

$$\lim_{n\to\infty} \frac{a^n - b^n}{a^n + b^n}.$$

(Here a, b are arbitrary real constants; you may have to treat cases.)

Solution: We deal with some boundary cases first.

- If $a = b = 0$, none of the terms of the sequence is defined, and the limit doesn't exist.
- If $a = -b \neq 0$ then every second term of the sequence is undefined, and the limit doesn't exist.
- If $a = b \neq 0$, all terms are 0 and the limit is 0.

We have dealt with the lines (in the a-b plane) $a = b$ and $a = -b$. Removing those lines, the plane breaks into 4 connected regions:

- $a > 0$, $|b| < a$. Here
$$\frac{a^n - b^n}{a^n + b^n} = \frac{1 - (b/a)^n}{1 + (b/a)^n} \to \frac{1-0}{1+0} = 1 \quad \text{as } n \to \infty.$$
- $a < 0$, $|b| < a$. Here
$$\frac{a^n - b^n}{a^n + b^n} = \frac{1 - (b/a)^n}{1 + (b/a)^n} \to \frac{1-0}{1+0} = 1 \quad \text{as } n \to \infty.$$
- $b > 0$, $|a| < b$. Here
$$\frac{a^n - b^n}{a^n + b^n} = \frac{(a/b)^n - 1}{(a/b)^n + 1} \to \frac{0-1}{0+1} = -1 \quad \text{as } n \to \infty.$$
- $b < 0$, $|a| < b$. Here
$$\frac{a^n - b^n}{a^n + b^n} = \frac{(a/b)^n - 1}{(a/b)^n + 1} \to \frac{0-1}{0+1} = -1 \quad \text{as } n \to \infty.$$

This covers all possibilities.

(b)

$$\lim_{n\to\infty} (n - \sqrt{n-a}\sqrt{n-b}).$$

(Again, a, b are arbitrary real constants; you may have to treat cases.)

Solution: Here cases should not be necessary.

$$\begin{aligned}
n - \sqrt{n-a}\sqrt{n-b} &= \frac{(n - \sqrt{n-a}\sqrt{n-b})(n + \sqrt{n-a}\sqrt{n-b})}{n + \sqrt{n-a}\sqrt{n-b}} \\
&= \frac{n^2 - (n-a)(n-b)}{n + \sqrt{n-a}\sqrt{n-b}} \\
&= \frac{(a+b)n - ab}{n + \sqrt{n-a}\sqrt{n-b}} \\
&= \frac{(a+b) - ab/n}{1 + \sqrt{1-a/n}\sqrt{1-b/n}}
\end{aligned}$$

The function
$$x \mapsto \frac{(a+b)-x}{1+\sqrt{1-ax}\sqrt{1-bx}}$$
is continuous at 0, and takes value $(a+b)/2$ there. Since $(1/n) \to 0$ as $n \to \infty$, we conclude from the continuity theorem for sequences that
$$\lim_{n\to\infty}(n-\sqrt{n-a}\sqrt{n-b}) = \frac{a+b}{2}$$
(and the argument was valid for all a, b).

6. **Note**: the common limit of the two sequences in this question is referred to as the *arithmetic-geometric mean* of a_1, b_1. It has a nice wikipedia page (google it).

   Define two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ recursively by $0 < a_1 < b_1$ (some arbitrary reals) and for $n \geq 1$
$$a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

   (a) Prove that $b_n \geq a_n$ for all $n \geq 1$.
   Solution: For $n = 1$, the claim is given. For $n > 1$, using the dual recurrence,
$$\begin{aligned} b_n \geq a_n \quad &\iff \quad \frac{a_{n-1}+b_{n-1}}{2} \geq \sqrt{a_{n-1}b_{n-1}} \\ &\iff \quad (a_{n-1}+b_{n-1})^2 \geq 4a_{n-1}b_{n-1} \\ &\iff \quad a_{n-1}^2 + 2a_{n-1}b_{n-1} + b_{n-1}^2 \geq 4a_{n-1}b_{n-1} \\ &\iff \quad a_{n-1}^2 - 2a_{n-1}b_{n-1} + b_{n-1}^2 \geq 0 \\ &\iff \quad (a_{n-1}-b_{n-1})^2 \geq 0, \end{aligned}$$
   which is true.

   (b) Prove that $(a_n)_{n=1}^{\infty}$ is non-decreasing.
   Solution: For each $n \geq 1$,
$$\begin{aligned} a_{n+1} \geq a_n \quad &\iff \quad \sqrt{a_n b_n} \geq a_n \\ &\iff \quad b_n \geq a_n, \end{aligned}$$
   which we have proven. So $(a_n)_{n=1}^{\infty}$ is non-decreasing.

   (c) Prove that $(b_n)_{n=1}^{\infty}$ is non-increasing.
   Solution: For each $n \geq 1$,
$$\begin{aligned} b_{n+1} \leq b_n \quad &\iff \quad \frac{a_n+b_n}{2} \leq b_n \\ &\iff \quad a_n \leq b_n, \end{aligned}$$
   which we have proven. So $(b_n)_{n=1}^{\infty}$ is non-increasing.

(d) Explain why $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ both converge to finite limits.

Solution: First, $(b_n)_{n=1}^{\infty}$ is non-increasing, and bounded below by 0, so converges to some limit, M say.

We have $a_n \leq b_n \leq b_1$, so $(a_n)_{n=1}^{\infty}$ is non-decreasing, and bounded above, by b_1; so it converges to some finite limit, L say.

(e) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Solution: We have $(a_n b_n) \to LM$ (by a basic limit theorem). Since the square root function is continuous at LM, it follows that $(\sqrt{a_n b_n}) \to \sqrt{LM}$. But $(\sqrt{a_n b_n}) = (a_{n+1}) \to L$. So $L = \sqrt{LM}$, and $L = M$.