Math 10860, Honors Calculus 2

Midterm 2 — solutions

Friday April 17

1. (a) (5 points) Suppose $(a_n)_{n=1}^{\infty}$ converges to limit L, and $(b_n)_{n=1}^{\infty}$ converges to limit $-L$. Carefully show that $(a_n+b_n)_{n=1}^{\infty}$ converges to limit 0.

   Solution: Given $\varepsilon > 0$, there is N_1 such that $n > N_1$ implies $|a_n - L| < \varepsilon/2$, and there is N_2 such that $n > N_2$ implies $|b_n - (-L)| = |b_n + L| < \varepsilon/2$. For $n > \max\{N_1, N_2\}$, have

   $$|a_n + b_n - 0| = |(a_n - L) + (b_n + L)| \leq |a_n - L| + |b_n + L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   Since $\varepsilon > 0$ was arbitrary, it follows that $(a_n + b_n)_{n=1}^{\infty}$ converges to limit 0.

   (b) (5 points) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence that converges to a finite limit. Set $b_n = \frac{a_n^2}{2+a_n^2}$. Prove that $(b_n)_{n=1}^{\infty}$ converges to a finite limit.

   Solution: Let L be the limit of (a_n). The function $f(x) = \frac{x^2}{2+x^2}$ is continuous at L (it doesn't matter whether L is positive, negative, or 0), so the sequence $(f(a_n))$ converges to the finite limit $\frac{L^2}{2+L^2}$. But $(f(a_n)) = (b_n)$, so we are done.
   Alternative solution: use the various basic properties of limits that we stated in lectures (and left as exercises): $a_n \to L$ so

   - $a_n^2 \to L^2$
   - $2 + a_n^2 \to 2 + L^2 \neq 0$
   - $\frac{L^2}{2+L^2} \to \frac{L^2}{2+L^2}$.

2. (a) (5 points) Compute $\int \sqrt{x} \log x \, dx$.

   Solution: Use integration by parts, with $u = \log x$ (so $du = dx/x$) and $dv = \sqrt{x} dx$ (so $v = (2/3)x^{3/2}$) to get

   $$\int \sqrt{x} \log x \, dx = \frac{2(\log x)x^{3/2}}{3} - \frac{2}{3}\int \sqrt{x} \, dx = \frac{2(\log x)x^{3/2}}{3} - \frac{4x^{3/2}}{9}.$$

   (b) (5 points) For most of the credit, use a substitution (or substitutions) to reduce to an integral of a rational function; for full credit complete the integration.

   $$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$$

Solution: Let $u = x^{1/6}$, so $x = u^6$, $dx = 6u^5 du$, and $\sqrt{x} + \sqrt[3]{x} = u^3 + u^2$, leading to

$$
\begin{aligned}
\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= 6 \int \frac{u^5 \, du}{u^3 + u^2} \\
&= 6 \int \frac{u^3 \, du}{u + 1} \\
&= 6 \int \left(u^2 - u + 1 - \frac{1}{u+1} \right) \\
&= 6 \left(\frac{u^3}{3} - \frac{u^2}{2} + u - \log|u+1| \right) \\
&= 6 \left(\frac{\sqrt{x}}{3} - \frac{x^{1/3}}{2} + x^{1/6} - \log(1 + x^{1/6}) \right).
\end{aligned}
$$

(No need for absolute value inside the log in the last line — in the domain of the integrand ($x > 0$), $1 + x^{1/6}$ is positive.)

3. Set $a_1 = 1$ and for $n \geq 1$, set

$$a_{n+1} = \sqrt{6 + a_n}.$$

(a) (4 points) Prove that $a_n \leq 3$ always.

Solution: (Had we not been given the hint that we should show an upper bound of 3, we could have guessed that this was the right choice for an upper bound. We could have guessed that if L is the limit, then $L = \sqrt{6+L}$, or $L^2 - L - 6 = 0$, or $(L+2)(L-3) = 0$, or $L =$ either -2 or 3; but since L should be positive, we could have guessed that $L = 3$, and then attempted to show that 3 is an upper bound).

We'll prove that $a_n \leq 3$ for all n, by induction on n, with the base case $n = 1$ already given. For $n \geq 1$, we have

$$
\begin{aligned}
a_{n+1} \leq 3 &\iff \sqrt{6 + a_n} \leq 3 \\
&\iff 6 + a_n \leq 9 \\
&\iff a_n \leq 3,
\end{aligned}
$$

which is the induction hypothesis, so we are done by induction.

(b) (4 points) Prove that $(a_n)_{n=1}^{\infty}$ is non-decreasing.

Solution: For $n \geq 1$,

$$
\begin{aligned}
a_{n+1} \geq a_n &\iff \sqrt{6 + a_n} \geq a_n \\
&\iff 6 + a_n \geq a_n^2 \\
&\iff 0 \geq a_n^2 - a_n - 6 \\
&\iff 0 \geq (a_n + 2)(a_n - 3).
\end{aligned}
$$

This is true; since $a_n \in [0,3]$ (by part (a)) we have $a_n + 2 \geq 0$ and $a_n - 3 \leq 0$ so $(a_n+2)(a_n-3) \leq 0$. This shows that $(a_n)_{n=1}^{\infty}$ is non-decreasing.

(c) (4 points) Explain why $(a_n)_{n=1}^{\infty}$ converges to a limit ℓ (just state the result we have proven that allows this to be concluded), and calculate ℓ (with brief justification).

Solution: (a_n) is non-decreasing and bounded above, so by a theorem from lectures it converges to a limit (which happens to be the supremum of the a_n). Suppose the limit is ℓ. Since $f(x) = \sqrt{6+x}$ is continuous at ℓ (certainly $\ell \geq 0$), it follows from the fundamental continuity theorem that $(f(a_n))$ converges to $f(\ell) = \sqrt{6+\ell}$. But $(f(a_n)) = (a_{n+1})$, which converges to ℓ, so by uniqueness of limits, we get $\ell = \sqrt{6+\ell}$. The only positive solution to this is $\ell = 3$, so that is the limit.

4. The Taylor polynomial of degree $2n$ of cos at 0 is

$$P_{2n,0,\cos}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Let $f(x) = \cos(x^2)$. It seems quite plausible that $P_{4n,0,f}(x)$, the Taylor polynomial of degree $4n$ of f at 0, is $P_{2n,0,\cos}(x^2)$, or

$$1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!}.$$

(a) (4 points) Use what you know about Taylor polynomials to show that the above polynomial is indeed $P_{4n,0,f}(x)$. **Hint, so you don't set off on the wrong path**: this question is about a Taylor polynomial of a *fixed* degree; it is *not* about what happens as n goes to infinity.

Solution: We know (from a theorem in lectures) that $P_{2n,0,\cos}(x)$ agrees to order $2n$ with cos at 0, meaning that

$$\lim_{x \to 0} \frac{P_{2n,0,\cos}(x) - \cos x}{x^{2n}} = 0.$$

But as $x \to 0$, so also does $x^2 \to 0$, so if we replace x with x^2 in the above expression, that limit is still 0 as $x \to 0$. In other words,

$$\lim_{x \to 0} \frac{P_{2n,0,\cos}(x^2) - \cos(x^2)}{x^{4n}} = 0.$$

This says that $P_{2n,0,\cos}(x^2)$ agrees to order $4n$ with $\cos(x^2)$ at 0, and so (again by a result from lectures),

$$P_{2n,0,\cos}(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!}$$

is the Taylor polynomial of degree $4n$ of $\cos(x^2)$ at 0.

(b) (2 points) Using the result of part (a) (or otherwise, but I wouldn't advise that!) find the 100th derivative of $f(x) = \cos(x^2)$ at 0, and the 102nd.

Solution: The coefficient of x^m in the Taylor polynomial of $f(x) = \cos(x^2)$ at 0 is

$$\frac{f^{(m)}(0)}{m!}.$$

From part (a), the coefficient of x^{100} is $\frac{-1}{50!}$, so

$$\frac{f^{(100)}(0)}{100!} = \frac{-1}{50!},$$

and so

$$f^{(100)}(0) = \frac{-100!}{50!}.$$

Also from part (a), the coefficient of x^{102} is 0, so

$$f^{(102)}(0) = 0.$$

An extra credit problem (2 points): For each real x find

$$\lim_{n\to\infty}\left(\lim_{k\to\infty}(\cos(n!\pi x))^{2k}\right).$$

(You are very familiar with the function that sends x to the above limit).

Solution: Suppose x is rational. Then for all large enough n, $n!\pi x$ is an integer multiple of π, and so $\cos(n!\pi x) = \pm 1$, so $(\cos(n!\pi x))^{2k} = 1$. So for all large enough n, $\lim_{k\to\infty}(\cos(n!\pi x))^{2k} = 1$, and so

$$\lim_{n\to\infty}\left(\lim_{k\to\infty}(\cos(n!\pi x))^{2k}\right) = 1.$$

Now suppose x is irrational. Then, for each fixed n, $n!\pi x$ is not an integer multiple of π (that would make π rational). So $|\cos(n!\pi x)| < 1$ and $0 \leq \cos^2(n!\pi x) < 1$. So, for fixed irrational x and natural number n, $\lim_{k\to\infty}(\cos(n!\pi x))^{2k} = 0$. It follows that in this case

$$\lim_{n\to\infty}\left(\lim_{k\to\infty}(\cos(n!\pi x))^{2k}\right) = 0.$$

So: the function that sends x to the above limit is the Dirichlet function!