Math 10860, Honors Calculus 2

Midterm 2 **NAME**:

Friday April 17

Instructions

- Attempt all questions for full credit.
- Please give complete and clear answers, et cetera, et cetera. (You know the drill by now. But I should add — this will be my first time e-grading. Please help me out by writing neatly!).
- Do the exam on your own paper. Put your name on top of the first page. Start each new question **on a new page**. Write the question number clearly.
- When you are done, scan all the pages, in order (question 1 first, then question 2, and so on), to a single pdf file, and email it back to me at `math10860homework at gmail.com`. You should return the exam to me within 90 minutes of receiving it, or risk losing points.
- It is ok to consult your notes, the textbook, and the course notes; but please avoid seeking out resources beyond these. And don't talk with anyone about the problems, while you are doing them! (Exception: it is ok to talk to your cat and/or dog.)

Peanuts by Charles Schultz

1 Problems (38 points total)

1. (a) (5 points) Suppose $(a_n)_{n=1}^{\infty}$ converges to limit L, and $(b_n)_{n=1}^{\infty}$ converges to limit $-L$. Carefully show that $(a_n + b_n)_{n=1}^{\infty}$ converges to limit 0.

   (b) (5 points) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence that converges to a finite limit. Set $b_n = \frac{a_n^2}{2+a_n^2}$. Prove that $(b_n)_{n=1}^{\infty}$ converges to a finite limit.

2. (a) (5 points) Compute $\int \sqrt{x} \log x \, dx$.

   (b) (5 points) For most of the credit, use a substitution (or substitutions) to reduce to an integral of a rational function; for full credit complete the integration.

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$$

3. Set $a_1 = 1$ and for $n \geq 1$, set

$$a_{n+1} = \sqrt{6 + a_n}.$$

   (a) (4 points) Prove that $a_n \leq 3$ always.

   (b) (4 points) Prove that $(a_n)_{n=1}^{\infty}$ is non-decreasing.

   (c) (4 points) Explain why $(a_n)_{n=1}^{\infty}$ converges to a limit ℓ (just state the result we have proven that allows this to be concluded), and calculate ℓ (with brief justification).

4. The Taylor polynomial of degree $2n$ of cos at 0 is

$$P_{2n,0,\cos}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

   Let $f(x) = \cos(x^2)$. It seems quite plausible that $P_{4n,0,f}(x)$, the Taylor polynomial of degree $4n$ of f at 0, is $P_{2n,0,\cos}(x^2)$, or

$$1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!}.$$

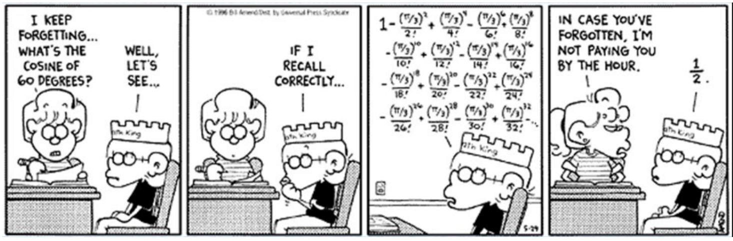
   (a) (4 points) Use what you know about Taylor polynomials to show that the above polynomial is indeed $P_{4n,0,f}(x)$. **Hint, so you don't set off on the wrong path**: this question is about a Taylor polynomial of a *fixed* degree; it is *not* about what happens as n goes to infinity.

   (b) (2 points) Using the result of part (a) (or otherwise, but I wouldn't advise that!) find the 100th derivative of $f(x) = \cos(x^2)$ at 0, and the 102nd.

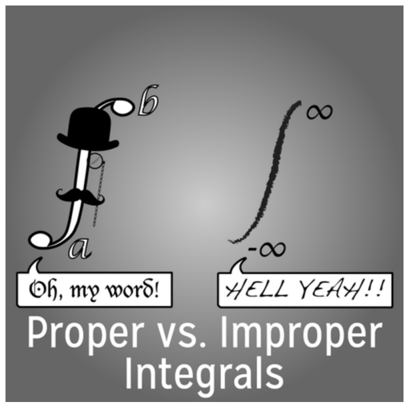
An extra credit problem (2 points): For each real x find

$$\lim_{n\to\infty} \left(\lim_{k\to\infty} (\cos(n!\pi x))^{2k} \right).$$

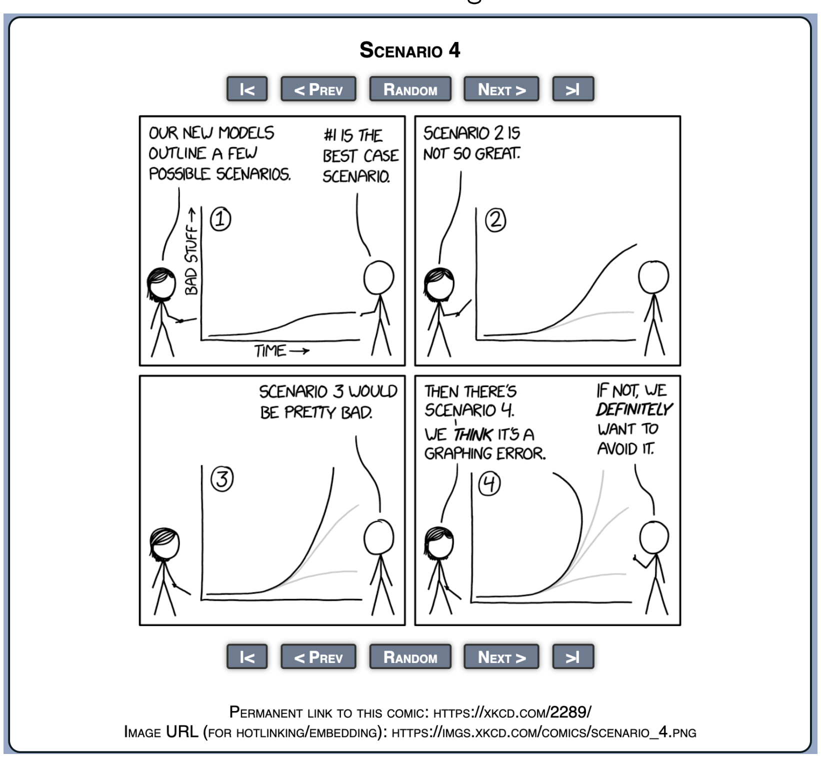
(You are very familiar with the function that sends x to the above limit).



Foxtrot by Bill Amend



Found at `https://lifethroughamathematicianseyes.wordpress.com/2015/10/21/fun-with-integrals/`



xkcd by Randall Munroe