Math 10860, Honors Calculus 2

Quiz 10, Thursday April 23

Solutions

1. Give a complete statement of the ratio test for series convergence.

   Solution: If (a_n) is a sequence of non-negative terms, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and equals r (acceptable: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$), then

   - if $r > 1$, (a_n) is not summable (or, equally acceptable, $\sum_{n=1}^{\infty} a_n$ does not converge, or is not finite, or does not exist, or equals infinity, or sums to infinity);
   - if $r < 1$, (a_n) is summable (or, equally acceptable, $\sum_{n=1}^{\infty} a_n$ converges, or is finite, or exists); and
   - if $r = 1$, no conclusion about the summability of (a_n) (or, equally acceptable, about the convergence or otherwise, or finiteness or otherwise, of $\sum_{n=1}^{\infty} a_n$) can be reached.

   Equally acceptable is a statement about the absolute summability of (a_n) in terms of $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$.

2. Determine for which real x does the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1}$ converge. *Briefly* justify your various assertions.

   Hint: Consider cases, the first of which should be very quick.

   Case 1 $x > 1$ and $x < -1$.

   Case 2 $0 \leq x \leq 1$ (note that the terms alternate in sign in this regime).

   Case 3 $-1 < x < 0$ (note that all terms are *positive* in this and the next regime).

   Case 4 $x = -1$.

   Solution:

   Case 1 $x > 1$ and $x < -1$: the nth term does not go to zero, so the series does not converge for any of these values.

   Case 2 $0 \leq x \leq 1$: here the series is

$$1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \cdots .$$

The sequence $(x^n/(2n+1))$ is decreasing: for $x > 0$,

$$\left(\frac{x^{n+1}}{2n+3} < \frac{x^n}{2n+1} \right) \iff \left(x < \frac{2n+3}{2n+1} \right),$$

which is certainly true for $x \leq 1$. Also, the sequence goes to 0 as $n \to \infty$, so by the Leibniz alternating series test the series converges.

Case 3 $-1 < x < 0$. Note that all terms are *positive* in this and the next regime, so the series is $\sum_{n=0}^{\infty} \frac{|x|^n}{2n+1}$. We use the ratio test on this series:

$$\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}(2n+1)}{(2n+3)|x|^n} = \frac{|x|(2n+2)}{2n+3} \to |x| < 1 \quad \text{as } n \to \infty,$$

so by the ratio test, the series converges.

Case 4 $x = -1$: If $x = -1$ the series becomes

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots,$$

which does not converge, for example by the limit comparison test, comparing with the Harmonic series (the nth term of the Harmonic series is $1/n$, and the n term of the present series is $1/(2n-1)$, so the limit of the ratio $1/2$).

In summary, $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2n+1}$ converges for $x \in (-1, 1]$ and nowhere else.

Note that one could also consider a case $-1 < x < 1$, and use the ratio test to show *absolute* convergence, and thus convergence, in this regime; but this wouldn't safe much time, as one would still have to do Leibniz' test at $x = 1$.