Math 10860, Honors Calculus 2

Quiz 2, Thursday January 30

Solutions

1. Let A be a non-empty set, bounded both above and below. Let $M = \sup A$ and $m = \inf A$. Let $-A = \{-x : x \in A\}$. What are $\sup(-A)$ and $\inf(-A)$? Justify *one* of your two claims carefully.

   Solution: We claim that $\sup(-A) = -m$ and $\inf(-A) = -M$. To prove the first of these, namely $\sup(-A) = -m$, we have to two two things: first, show that $-m$ is an upper bound for $-A$, and second, show that it a *least* upper bound.

   - $-m$ **is an upper bound for** $-A$: For each $x \in A$ we have $m \le x$, so $-m \ge -x$. This says that $-m \ge y$ for all $y \in -A$ (the elements of $-A$ are exactly numbers of the form $-x$ for $x \in A$), so $-m$ is an upper bound for $-A$.
   - $-m$ **is a *least* upper bound for** $-A$: Let t be any upper bound for $-A$. We have $t \ge y$ for all $y \in -A$, so $t \ge -x$ for all $x \in A$, so $-t \le x$ for all $x \in A$, so $-t$ is a lower bound for A, so $-t \le m$ (since m is a *greatest* lower bound), so $t \ge -m$. Hence $-m$ is indeed a *least* upper bound for $-A$.

   The proof that $\inf(-A) = -M$ is almost identical.

2. Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Let P be a partition of $[a, b]$. Prove *one* of the following two claims (they are both true, and have almost identical proofs. It might be helpful to use part 1):

   (a) $U(-f, P) = -L(f, P)$
   (b) $L(-f, P) = -U(f, P)$.

   Solution: We just prove part (a). We assume $P = (t_0, \ldots, t_n)$. In the second line below, we use the fact that

   $$\{-f(x) : x \in [t_{i-1}, t_i]\} = -\{f(x) : x \in [t_{i-1}, t_i]\},$$

   so that, by part 1,

   $$\sup\{-f(x) : x \in [t_{i-1}, t_i]\} = -\{f(x) : x \in [t_{i-1}, t_i]\} = \sup\left(-\{f(x) : x \in [t_{i-1}, t_i]\}\right) = -\inf\{f(x) : x \in [t_{i-1}, t_i]\}.$$

   We have

   $$\begin{aligned} U(-f, P) &= \sum_{i=1}^{n} \sup\{-f(x) : x \in [t_{i-1}, t_i]\}\Delta_i \\ &= \sum_{i=1}^{n} -\inf\{f(x) : x \in [t_{i-1}, t_i]\}\Delta_i \\ &= -\sum_{i=1}^{n} \inf\{f(x) : x \in [t_{i-1}, t_i]\}\Delta_i \\ &= -L(f, P). \end{aligned}$$

   The proof that $L(-f, P) = -U(f, P)$ is almost identical.

3. Suppose f as in part 2 is integrable. Use the result of part 2 to show that $-f$ is also integrable.

   Solution: Let $\varepsilon > 0$ be given. Since f is integrable there is a partition P of $[a, b]$ for which $U(f, P) - L(f, P) < \varepsilon$. For this partition P we have (by part 2)

   $$U(-f, P) - L(-f, P) = -L(f, P) - (-U(f, P)) = U(f, P) - L(f, P) < \varepsilon.$$

   Since $\varepsilon > 0$ was arbitrary, this shows that $-f$ is integrable.

   Note: The question did not ask this, but here is a very quick proof that $\int_a^b -f = -\int_a^b f$. By hypothesis f is integrable, and we have just shown that $-f$ is integrable, so (by the addition result) we know $f + (-f)$ is integrable, and $\int_a^b (f + (-f)) = \int_a^b f + \int_a^b -f$. But $f + (-f) = 0$, so $\int_a^b (f + (-f)) = 0$. It follows that

   $$\int_a^b f + \int_a^b -f = 0,$$

   so $\int_a^b -f = -\int_a^b f$.