# Math 10860, Honors Calculus 2

Quiz 4, Thursday February 13

Solutions

1. Suppose that $f$ is a one-to-one function, that is differentiable everywhere on its domain, with derivative never zero. Suppose also that there is a function $F$ with $F' = f$.

   Set $G(x) = xf^{-1}(x) - F(f^{-1}(x))$. Verify that $G'(x) = f^{-1}(x)$. (**Remark**: so, if we know an antiderivative of $f$, we also know an antiderivative of $f^{-1}$.)

   **Solution**:

$$\begin{aligned} G'(x) &= x(f^{-1})'(x) + f^{-1}(x) - F'(f^{-1}(x))(f^{-1})'(x) \quad \text{(product rule, chain rule)} \\ &= \frac{x}{f'(f^{-1}(x))} + f^{-1}(x) - \frac{f(f^{-1}(x))}{f'(f^{-1}(x))} \quad (F' = f \text{ and } f' \text{ never zero}) \\ &= \frac{x}{f'(f^{-1}(x))} + f^{-1}(x) - \frac{x}{f'(f^{-1}(x))} \quad (f(f^{-1}(x)) = x) \\ &= f^{-1}(x) \end{aligned}$$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, invertible and satisfies $f^{-1} = f$. Prove that $f$ has a fixed point (a number $x$ such that $f(x) = x$).

   **Solution**: Given $a \in \mathbb{R}$, we either have $f(a) = a$, in which case we are done, or we have $f(a) \neq a$. In the latter case, either $f(a) < a$ or $f(a) > a$. In either case, set $f(a) = b$.

   We have $f^{-1}(b) = a$, but since $f^{-1} = f$ this says that $f(b) = a$.

   Now consider the function $g$ given by $g(x) = f(x) - x$. On the closed interval $[a, b]$, $g$ takes on both negative and positive values (if $b < a$ then $g(a) = f(a) - a = b - a < 0$ and $g(b) = f(b) - b = a - b > 0$, and similarly if $b > a$ then $g(a) > 0$ and $g(b) < 0$). Since $g$ is continuous, the intermediate value theorem ensures that there is $x \in [a, b]$ with $g(x) = 0$ so $f(x) = x$.