Math 10860, Honors Calculus 2

Quiz 6, Thursday February 27

Solutions

1. When we began discussing the trigonometric functions, we gave a precise definition of the function cos on the domain $[0, \pi]$. State that definition.[1]

   Solution: For $x \in [0, \pi]$, $\cos x$ is defined to be $A^{-1}(x/2)$, where
   $$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\, dt.$$
   (Equivalently, with A as defined above, $\cos x$ is that unique number ? such that $A(?) = x/2$.)

2. From the angle-summation formulae (together with the basic Phythagorean identity connecting sin and cos), other useful formulae can be deduced.

   (a) Verify that[2]
   $$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

   Solution: We recall the angle-summation formulae: $\cos(x+y) = \cos x \cos y - \sin x \sin y$ and $\sin(x+y) = \sin x \cos y + \cos x \sin y$. Applying the first with $x = y = \theta$ we get
   $$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$
   Also
   $$1 = \cos^2\theta + \sin^2\theta.$$
   Adding these two and dividing by 2 gives the desired identity. (Subtracting and dividing by 2 gives the one mentioned in the footnote.)

   (b) Verify that[3]
   $$\tan(t/2) = \frac{\sin t}{1+\cos t}.$$

   Solution: One solution goes like this: we have $\cos t = \cos^2(t/2) - \sin^2(t/2)$ and $\sin t = 2\sin(t/2)\cos(t/2)$ (the second from the sin summation formula), so
   $$\begin{aligned}\frac{\sin t}{1+\cos t} &= \frac{2\sin(t/2)\cos(t/2)}{(\cos^2(t/2)+\sin^2(t/2)) + (\cos^2(t/2)-\sin^2(t/2))}\\ &= \frac{2\sin(t/2)\cos(t/2)}{2\cos^2(t/2)}\\ &= \frac{\sin(t/2)}{\cos(t/2)}\\ &= \tan(t/2),\end{aligned}$$

[1] You don't need to argue that it is a meaningful definition.
[2] There's also $\sin^2\theta = \frac{1-\cos\theta}{2}$, which is worth remembering.
[3] This also equals $\frac{1+\cos t}{\sin t}$

the division in the last step valid because $\cos(t/2) \neq 0$ (since implicitly $\tan(t/2)$ is defined). (The identity in the footnote can be proved similarly).

Another solution might proceed by using

$$\cos(t/2) = \sqrt{\frac{1+\cos t}{2}}, \qquad \sin(t/2) = \sqrt{\frac{1-\cos t}{2}},$$

but this requires care, as in reality

$$\cos(t/2) = \pm\sqrt{\frac{1+\cos t}{2}}, \qquad \sin(t/2) = \pm\sqrt{\frac{1-\cos t}{2}},$$

and we have to break into cases, since the sign in front of the square root depends on the particular t under consideration.