Math 10860, Honors Calculus 2

Quiz 7

Solutions

1. Use the substitution $u = \sqrt{1+\sqrt{x}}$ to find
$$\int \frac{dx}{\sqrt{1+\sqrt{x}}}.$$

Solution: With $u = \sqrt{1+\sqrt{x}}$, have $\sqrt{x} = u^2 - 1$ and
$$du = \frac{dx}{4\sqrt{x}\sqrt{1+\sqrt{x}}} = \frac{dx}{4(u^2-1)u}$$
so
$$dx = (4u^3 - 4u)du$$
and so
$$\begin{aligned} \int \frac{dx}{\sqrt{1+\sqrt{x}}} &= \int \frac{4u^3-4u}{u} du \\ &= \int (4u^2-4)du \\ &= \frac{4}{3}u^3 - 4u \\ &= \frac{4}{3}(1+\sqrt{x})\sqrt{1+\sqrt{x}} - 4\sqrt{1+\sqrt{x}} \\ &= \frac{4}{3}(\sqrt{x}-2)\sqrt{1+\sqrt{x}}. \end{aligned}$$

2. Using an appropriate substitution, reduce
$$\int \frac{x^2}{\sqrt{x^2+1}}\, dx$$
to an integral that only involves trigonometric functions.

Solution: The obvious substitution is $x = \tan t$. This gives $dx = \sec^2 t\; dt$, and
$$\sqrt{x^2+1} = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = \sec t.$$
Note that the domain of $\sqrt{x^2+1}$ is all real x, so $t = \arctan(x)$ varies over $(-\pi/2, \pi/2)$ (the full range of arctan); on this interval sec is positive, so it is valid to write
$$\sqrt{\sec^2 t} = \sec t$$
(we do not need to write $\sqrt{\sec^2 t} = |\sec t|$). So
$$\int \frac{x^2}{\sqrt{x^2+1}}\, dx = \int \frac{\tan^2 x \sec^2 x}{\sec x}\, dx = \int \tan^2 x \sec x\; dx = \int \frac{\sin^2 x}{\cos^3 x}\, dx.$$
(Any of these expressions, or indeed any other equivalent expression, is fine here.)