Math 10860, Honors Calculus 2

Quiz 8

Solutions

1. Taylor's theorem with Lagrange remainder term says that given a function f, a real a, a natural number n, and a real x, if certain hypotheses are satisfied then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{2}(x-a)^n + R_{n,a,f}(x),$$

with $R(n,a,f)(x)$ taking a certain form. State the necessary hypotheses, and the exact form that $R(n,a,f)(x)$ takes.

Solution: The necessary hypotheses, as stated in the version of Taylor's theorem with Lagrange remainder form proved in the lectures and stated in the notes as the *weak* form of the theorem, are that

- all the derivatives of f up to and including the $(n+1)$th derivative exist on an interval that includes a and x, and
- $f^{(n+1)}$ is continuous on the interval.

The exact form of the remainder term is

$$R_{n,a,f}(x) = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}$$

where c is some number that lies between a and x.

In the notes, there is also a *strong* form of the theorem mentioned (in a footnote), that drops that condition that $f^{(n+1)}$ is continuous, and just requires that it exists.

2. Show that $x + x^2/2$ agrees to order 2 with $\log(x^2+x+1)$ at 0.

Solution: A slick way to do this is to compute the Taylor polynomial of degree 2 of $f(x) = \log(x^2+x+1)$ at $a = 0$, and check that it is $x + x^2/2$. Then the fact (presented, with proof, in notes and lectures) that

> the *unique* polynomial of degree 2 that agrees to order 2 with f is the Taylor polynomial $P_{2,a,f}(x)$

gives the result.

The more prosaic solution:

$$\begin{aligned}\lim_{x\to 0}\frac{x + x^2/2 - \log(x^2+x+1)}{x^2} &= \lim_{x\to 0}\frac{1 + x - \frac{2x+1}{x^2+x+1}}{2x} \\ &= \lim_{x\to 0}\frac{x^3+2x^2}{2x^2+2x+2} \\ &= 0,\end{aligned}$$

where in the first line we use L'Hôpital's rule, whose validity is contingent on the right-hand limit existing, which it does, as verified by the subsequent lines. Since the limit is 0, we see that $x + x^2/2$ agrees with $\log(x^2 + x_1)$ to order 2 at $a = 0$.

3. What is the highest order to which $x^7 + x^6$ and $x^7 - x^5$ agree at 0?

Solution: $(x^7 + x^6) - (x^7 - x^5) = x^6 + x^5$. We have

$$\lim_{x \to 0} \frac{x^6 + x^5}{x^4} = 0,$$

but $\lim_{x \to 0}(x^6 + x^5)/x^5 = 1$ and $\lim_{x \to 0}(x^6 + x^5)/x^n$ does not exist for any $n > 5$. So the highest order to which $x^7 + x^6$ and $x^7 - x^5$ agree at 0 is 4.