Math 10860, Honors Calculus 2

Quiz 9

Solutions

1. State the precise definition of "sequence $(a_n)_{n=1}^{\infty}$ converges to limit L as $n \to \infty$"

   Solution: $(a_n)_{n=1}^{\infty}$ converges to L as $n \to \infty$ if for all $\varepsilon > 0$ there exists N (or $N > 0$ or $N \geq 0$ or $\ldots$) such that $n > N$ (or $n \geq N$) implies $|a_n - L| < \varepsilon$.

2. Using the definition, show that $(1/\sqrt{n}) \to 0$ as $n \to \infty$.

   Solution: Let $\varepsilon > 0$ be given. Take $N = (1/\varepsilon)^2$ (or anything larger). Then $n > N$ implies $n > (1/\varepsilon)^2$ implies $1/\sqrt{n} < \varepsilon$ implies $|1/\sqrt{n} - 0| < \varepsilon$, as required. (Note everything in sight is positive).

3. Suppose $(a_n) \to L$ as $n \to \infty$, with $L > 0$. Prove that $(1/a_n) \to 1/L$ as $n \to \infty$.

   Solution: Given $\varepsilon > 0$ we want to find N such that $n > N$ implies

   $$\left| \frac{1}{a_n} - \frac{1}{L} \right| < \varepsilon,$$

   or equivalently

   $$\frac{|a_n - L|}{|a_n||L|} < \varepsilon.$$

   Now by $a_n \to L > 0$ we know there is N_1 such that $n > N_1$ implies $a_n > L/2$ so $1/(|a_n L|) < 2/L^2$; and there is N_2 such that $n > N_2$ implies $|a_n - L| < \varepsilon L^2/2$.

   Taking $N = \max\{N_1, N_2\}$ (or anything larger) we have that $n > N$ implies

   $$\left| \frac{1}{a_n} - \frac{1}{L} \right| = \frac{|a_n - L|}{|a_n||L|} < \left(\frac{\varepsilon L^2}{2} \right) \left(\frac{2}{L^2} \right) = \varepsilon,$$

   so $(1/a_n) \to 1/L$ as $n \to \infty$.