Math 10860, Honors Calculus 2

Final exam information

April 29, 2020

Basic organizational information

The final will be on

Monday May 9, 4.15pm-6.15pm (South Bend time).

In preparation for the final, I will have office hours as follows (the Friday time is definite; the others may change):

- Friday, 3-4.30
- Saturday, 2-3.30
- Monday, 11-12.30

all at the usual place — `https://notredame.zoom.us/j/844811786` (password 10860). Sarah will have office hours

- Sunday, 2-4

at `https://notredame.zoom.us/j/92657621624`.

The organization will be similar to that of the of the midterm. Check in at `https://notredame.zoom.us/j/844811786` sometime between 4 and 4.15, and I will share the exam with you. The exam must be returned to me by 6.30pm. You do not need to stay on the Zoom meeting while taking the exam.

Format and topics

The exam will be open-book, open-notes, but you should not consult any resources other than textbook and lecture notes. Because it will be open-book, it will be focussed on problems and things to prove (i.e., nothing that you can just take straight from the notes).

The exam is cumulative, but will probably skew a little more towards material from the end of the semester. In particular, *there will be a question on the material* *from the end of the semester, on uniform convergence and power series* (lectures from Friday April 24, Monday April 27 and Wednesday April 29, but not the last two clips from the last lecture). Since we haven't had homework on this material, at the end of this document I've put a few questions on this topic, and *I promise that one of the question on the final will be modeled very closely on the questions here.*

For the remaining material, I urge you to review old exams, homeworks and quizzes, to review the list of topics given below, and to talk to me and Sarah.

List of topics covered this semester

- Partitions, Darboux sums, definition of (Darboux) integral
- $U(f, P) - L(f, P) < \varepsilon$ criterion for integrability
- Example of integrable function with infinitely many discontinuities
- Splitting the interval of integration, linearity of integral, upper and lower bounds on integral in terms of upper and lower bounds on function
- Integrability closed under sums, scalar multiplication, multiplication, changing at finitely many values, taking absolute value, taking max, min, positive and negative parts, Cauchy-Schwarz inequality
- Uniform continuity, integrability of uniformly continuous functions, uniform continuity of continuous functions on closed intervals
- The fundamental theorems of calculus, parts 1 and 2
- Defining functions via $F(x) = \int_a^x f$, and manipulating the results (especially differentiating) when simple limits replaced by functions
- Improper integrals over infinite ranges, improper integrals of unbounded functions, comparison test for improper integrals, status of $\int_0^1 1/t^\alpha dt$ and $\int_1^\infty 1/t^\alpha dt$
- One-to-one and invertible functions, graphs of inverse functions
- Continuous one-to-one functions on an interval are monotone, determining range (so domain of inverse)
- Continuity and differentiability of inverse functions
- Defining and differentiating $f(x) = x^a$ for rational a
- Defining the logarithm function via an integral, and the exponential function as its inverse, basic properties of exp and log (algebraic identities, derivatives), definition of e, e^x, a^x

- Characterizing exp as solution to a differential equation
- Exponential functions grow faster than polynomial functions
- Informal (geometric) definition of $\sin, \cos$
- Definition of π, formal definitions of cos, sin via integration, basic properties of the trigonometric functions (identities, derivatives)
- Characterizing $\sin, \cos$ as solutions to a differential equation, angle addition formulae
- The hyperbolic functions — definition, explicit formulae, basic properties
- Primitives, or antiderivatives, definition and notation, unique* up to additive constants, linearity
- Common primitives ($\log, \exp$, trigonometric, hyperbolic and related)
- Integration by parts, definite and indefinite versions, reduction formulae
- Integration by substitution, general principle, definite and indefinite versions, using trigonometric substitutions
- Integrating products of powers of sin and cos, half-angle formulae
- Substitution $t = \tan(x/2)$
- Method of partial fractions
- Taylor polynomial of a function about a point, agreement to order n at a point, uniqueness of the Taylor polynomial
- Finding Taylor polynomial via expanding a series for derivative and then integrating, e.g., $\tan^{-1}$, log
- Taylor's theorem with integral remainder formula, proof via integration by parts, derivation of Lagrange remainder form assuming continuity of $(n+1)$st derivative
- Analyzing remainder term of Taylor series for exp, sin, cos, log, $\tan^{-1}$, estimating functions at points
- Factorial function grows faster than exponential
- Sequences, convergence, convergence to ∞ and $-\infty$
- Limits of sums, products, reciprocals scalar multiples of sequences, limits of rational functions

- Interaction between limits of sequences and limits of functions, limit of (a^n), limit of a continuous function of a sequence
- Bounded and monotone sequences, bounded monotone sequences converge, every sequence has a monotone subsequence, Bolzano-Weierstrass theorem
- Cauchy sequences, equivalence of being Cauchy and being convergent
- Series, summability, divergence of series, the harmonic series, the geometric series
- Closure of summable sequences under linear combinations, Cauchy criterion for summability, the vanishing condition, the boundedness criterion, the comparison test, the limit comparison test, the ratio test, the integral test, Leibniz' theorem on alternating series
- Absolute and conditional convergence, absolute convergence implies convergence, facts (but not proofs) about rearrangements of absolutely convergent sequences and rearrangements of conditionally convergent sequences
- Pointwise convergence of functions to limit, nuisance examples showing pointwise limit not well-behaved
- Uniform converge of functions on an interval to limit, preservation of continuity, limit of integral is integral of limit, conditions under which derivative of limit is limit of derivatives
- Power series, Taylor series, Maclaurin series, uniform convergence of power series below any point of convergence, radius of convergence, integrating and differentiation term-by-term inside radius of convergence

Problems on uniform convergence and power series

1. For each of these sequences $(f_n)_{n=1}^{\infty}$ of functions, decide whether the sequence tends pointwise to a limit function f on the given domain, and say what f is if it exists. Also, if the pointwise limit exists, determine if the convergence is uniform.

   (a) $f_n(x) = \sqrt[n]{x}$ on $[0, 1]$.

   (b) $f_n(x) = \frac{e^x}{x^n}$ on $(1, \infty)$.

   (c) $f_n(x) = x^n - x^{2n}$ on $[0, 1]$.

2. (a) By using an appropriate power series, evaluate

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}\pi^{2n}}{(2n)!}.$$

(b) By appropriate manipulation of the geometric series, evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$, $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{n3^n}$.

(c) Evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.

3. (a) The degree $2n+1$ Taylor polynomial of sin centered at 0 is

$$s_n(x) = \sum_{k=0}^{n} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

We already know that $s_n \to \sin$ *pointwise* as $n \to \infty$, for all $x \in \mathbb{R}$. Using what you know about remainder terms, show that $s_n \to \sin$ *uniformly* on the interval $[-x_0, x_0]$, for every $x_0 > 0$.

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find (with justification) a power series (centered at 0) that converges to f uniformly on every interval $[-x_0, x_0]$, $x_0 > 0$.

(c) With f as in the last part of this question, find $f^{(k)}(0)$ for every k.

(d) Write down (with justification) a series whose sum is $\int_0^1 \frac{\sin x}{x} dx$.