A A quick introduction to sets

The real numbers, that will be our main concern this semester, is a *set* of objects. Functions of the real numbers have associated with them two *sets* — their domain (the set of possible inputs) and their range (the set of possible outputs). A function itself will be defined as a *set* of ordered pairs. Sets are everywhere in mathematics, and so it will be important to have a good grasp on the standard notations for sets, and on the standard ways by which sets can be manipulated.

Formally defining what a set is is one of the concerns of logic, and goes well beyond the scope of this course. For us, a *set* will simply be a well-defined collection of objects — a collection for which there is an unambiguous test that determines whether something is in the collection of not. So, for example, the collection of good actors will not be a set (it's open to debate who is and isn't good), but the collection of best actor or actress Oscar winners from the last twenty years is a set.

A.1 Notation

We represent sets by putting the elements between braces; thus

$$A = \{1, 2, 3, 4, 5\}$$

is the set of all integers between 1 and 5 inclusive. We can list the elements like this only when the set has finitely many elements, and indeed practical considerations dictate that the set needs to be quite small to admit an explicit listing. (I would not like to have to list all the integers between 1 and $10^{10^{10}}$, for example.) We thus need compact ways to represent sets, and some of these ways are described in the next section.

Two sets A and B will be said to be equal if they have the same elements, that is, if for every x, if x is in A then it is in B, and if it is in B then it is in A. A consequence of this is that if we re-arrange the order in which the elements of a set are presented, we get the same set. So each of

$$\{1, 2, 3, 4, 5\}, \quad \{5, 4, 3, 2, 1\}, \quad \{2, 5, 4, 3, 1\}, \quad \{4, 3, 1, 2, 5\}$$

are the *same* set.

A set cannot contain a repeated element. We do not write $A = \{1, 1, 2, 3, 4, 5\}$. And so, although Hilary Swank has won two Oscars for best actress in the last twenty years, she would only be listed once in the set described in the last section. (There is such a thing as a *multiset* where repeated elements are allowed, but we won't think about this.)

The standard convention for representing sets

In calculus we work mainly with sets of real numbers. The most common notation/way to describe such a set is as follows:

$$S = \{x : p(x)\} \quad (\text{or } \{x | p(x)\})$$

where $p(x)$ is some predicate; the set S consists of all real numbers x such that $p(x)$ is true. The way to read this is

"S is the set of all x such that $p(x)$" (or, "such that $p(x)$ holds").

For example

$$\{x : x \geq 0\}$$

is the set of all non-negative numbers,

$$\{w : \text{the decimal expansion of } w \text{ contains no 3's}\}$$

describes a somewhat complicated set of real numbers, and

$$\{t : 3t^3 - 2t^2 - t > 1\}$$

describes a less complicated, but still hard to pin down, subset of the real numbers.

Sometimes we cheat a little and put an extra condition on the variable before the ":", to make things easier to write. For example, the domain of the function $g(y) = \sqrt{y}/(y-2)$ is all non-negative reals (negative reals don't have square roots), except 2 (we can't divide by 0), so we should write

$$\text{Domain}(S) = \{y : y \geq 0 \text{ and } y \neq 2\},$$

but it makes sense to write, slightly more compactly,

$$\text{Domain}(S) = \{y \geq 0 : y \neq 2\}.$$

The ellipsis notation

We sometimes describe a set by listing the first few elements, enough so that a pattern emerges, and then putting an ellipsis (a ...) to say "and so on". For example

$$\{2, 3, 5, 7, 11, \ldots\}$$

is *probably* the set of prime numbers. I say "probably", because there are plenty of reasonably natural sequences that begin 2, 3, 5, 7, 11, and are *not* the prime numbers. The amazing Online Encyclopedia of Integer Sequence, oeis.org (which is just what its name says) lists 956 such sequences, including the sequence of *palindromic* primes (prime numbers whose decimal expansion is a palindrome; the next is 101).

Because of these possible ambiguities, ellipsis notation should be used only when the context is absolutely clear. In the above example, something like

$$\{n : n \text{ is a palidromic prime}\}$$

should be preferred.

Ellipsis notation is sometimes used for a finite set. In this case, after the ellipsis there should be one or two terms, used to indicate where one should stop with the pattern. For example,

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{100}\right\}$$

probably indicates the reciprocals (multiplicative inverses) of all the natural numbers between 1 and 100; to make this absolutely clear one should write

$$\{1/n : 1 \leq n \leq 100,\ n \in \mathbb{N}\}.$$

Special sets of reals

There are some special sets of reals that occur so frequently, we give them special names. These are the *intervals* — sets of all numbers between two specified reals, with slightly different notation depending on whether the specified end points belong or don't belong to the set. Here's a list of all the incarnations that occur.

- $[a, b] = \{x : a \leq x \leq b\}$
- $[a, b) = \{x : a \leq x < b\}$
- $(a, b] = \{x : a < x \leq b\}$
- $(a, b) = \{x : a < x < b\}$
- $[a, \infty) = \{x : a \leq x\}$
- $(a, \infty) = \{x : a < x\}$
- $(-\infty, b] = \{x : x \leq b\}$
- $(-\infty, b) = \{x :< b\}$
- $(-\infty, \infty) = \mathbb{R}$.

Notice that a square bracket ("[" or "]") is used to indicate that the relevant endpoint is in the interval, and a round bracket ("(" or ")") is used to indicate that it is not. Notice also that we never put a "[" before $-\infty$ or a "]" after ∞; Neither $-\infty$ nor ∞ are numbers, merely notational symbols.

A.2 Manipulating sets

- It is possible for a set to contain no elements. We use the symbol $\emptyset$ to denote this *null* or *empty* set.

- We use the symbol "$\in$" to indicate membership of a set, so $x \in S$ indicates that x is an element of S. On the other side, $x \notin S$ indicates that x is *not* an element of S.
- When there is a clear universe U of all objects under discussion, we denote by S^c or S' the *complement* of S — the set of all elements in U that are not in S. So, for example, if it is absolutely clear that the universe of objects under discussion is the reals, then

  $$(0, \infty)^c = (-\infty, 0].$$

  If instead it is absolutely clear that the universe of objects under discussion is the set of non-negative reals, then

  $$(0, \infty)^c = \{0\} \quad \text{(the set containing the single element 0)}.$$

- If all the elements of a set A are also elements of a set B, we say that A is a *subset* of B and write $A \subseteq B$. For example, the set of prime numbers is a subset of the set of natural numbers. The two lines at the bottom of the symbol "$\subseteq$" are intended to convey that it is possible that $A = B$, that is, that A and B have exactly the same elements. In order words, any set is a subset of itself.

  If $A \subseteq B$ and $A \neq B$ (so there are some elements of B that are not elements of A) then A is said to be a *proper* subset of B, and this is sometimes written $\subsetneq$, or $\subsetneqq$, or $\varsubsetneqq$. It is also sometimes written $\subset$, but be warned that for many writers $A \subset B$ and $A \subseteq B$ are identical.

  To prove that $A \subseteq B$ it is necessary to prove the implication $A(x) \implies B(x)$ where $A(x)$ is the predicate $x \in A$ and $B(x)$ is the predicate $x \in B$. To prove that $A = B$, it is necessary to prove the equivalence $A(x) \iff B(x)$, which we know really requires two steps: showing $A(x) \implies B(x)$ ($A \subseteq B$) and $B(x) \implies A(x)$ ($B \subseteq A$).

  The empty set $\emptyset$ is a subset of every set.

  The collection of all subsets of a set S is called the *power set* of a set, written $\mathcal{P}(S)$. We will rarely use this.

A.3 Combining sets

There are a number of ways of combining old sets to form new ones.

- **Union**: The union of sets A and B, written $A \cup B$, is the set of all elements that are in either A or B, or perhaps both:

  $$A \cup B = \{x : (x \in A) \vee (x \in B)\}.$$

  For example, $[0, 1] \cup [1, 2) = [0, 2)$.

- **Intersection**: The intersection of sets A and B, written $A \cap B$, is the set of all elements that are in both A and B:

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}.$$

  For example, $[0, 1] \cap [1, 2) = \{1\}$.

- It is possible to take the intersection or union of arbitrarily many sets. The notation $\{A_i : i \in I\}$ is used to indicate that we have a family of sets, *indexed* by the set I: for each element i of I, there is a set A_i in our family. Often I is the set of natural numbers, and then we can write the family as

$$\{A_1, A_2, A_3, \ldots\}.$$

  The intersection of the sets in a family $\{A_i : i \in I\}$, written $\cap_{i \in I} A_i$, is the set of elements that are in all the A_i, while the union $\cup_{i \in I} A_i$ is the set of elements that are in at least one of the A_i.

  For example, if $I = \mathbb{N}$ and $A_i = (-i, i)$, then

$$\cup_{i \in I} A_i = \mathbb{R} \quad \text{and} \quad \cap_{i \in I} A_i = (-1, 1).$$

- The notation $A \setminus B$, sometimes written $A - B$, denotes the set of all elements that are in A but are not in B (the $-$ sign indicating that we have removed those elements). It is not necessary for B to be a subset of A for this to makes sense. So, for example

$$\{1, 2, 3, 4\} \setminus \{3, 4, 5, 6\} = \{1, 2\}.$$

- Notice that we always have the relations $A \setminus B \subseteq A$, $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$.

- The *Cartesian product* of two sets X and Y, denoted by $X \times Y$, is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. Note that when we describe a list of elements with round brackets "(" and ")" on either side, the order in which we present the list matters: (a, b) is not the same as (b, a) (unless $a = b$), whereas $\{a, b\} = \{b, a\}$ since both, as sets, have the same collection of elements. Also note that an ordered pair allows repetitions: $(3, 3)$ is a perfectly reasonable ordered pair.

A.4 The algebra of sets

The relations satisfied by union, intersection and complementation bear a striking resemblance to the relations between the logical operators of OR, AND and negation. [This is essentially for the following reason: given a universe of discourse U, and any predicate $p(x)$, we can associate a subset A of U via $A = \{x \in U : p(x) \text{ is true}\}$. Most of the logical equivalences that we have discussed between propositions have direct counterparts as equalities between the corresponding sets.]

We list the relations that hold between sets A, B and C that are all living inside a universe U. For comparison, we also list the corresponding relations that hold between propositions p, q and r. You should notice a direct correspondence between $\vee$ and $\cup$, between $\wedge$ and $\cap$, between negation and complementation, between T and U and between F and $\emptyset$. You should be able to come up with proofs of any/all of these identities.

Name of law	Equality/equalities	Equivalence(s)
Identity	$A \cap U = A$ $A \cup \emptyset = A$	$p \wedge T \iff p$ $p \vee F \iff p$
Domination	$A \cup U = U$ $A \cap \emptyset = \emptyset$	$p \vee T \iff T$ $p \wedge F \iff F$
Idempotent	$A \cup A = A$ $A \cap A = A$	$p \vee p \iff p$ $p \wedge p \iff p$
Double negation	$(A^c)^c = A$	$\neg(\neg p) \iff p$
Commutative	$A \cup B = B \cup A$ $A \cap B = B \cap A$	$p \vee q \iff q \vee p$ $p \wedge q \iff q \wedge p$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = (A \cap B) \cap C$	$(p \vee q) \vee r \iff p \vee (q \vee r)$ $(p \wedge q) \wedge r \iff (p \wedge q) \wedge r$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
De Morgan's	$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$	$\neg(p \wedge q) \iff (\neg p) \vee (\neg q)$ $\neg(p \vee q) \iff (\neg p) \wedge (\neg q)$
Absorption	$A \cap (A \cup B) = A$ $A \cup (A \cap B) = A$	$p \wedge (p \vee q) \iff p$ $p \vee (p \wedge q) \iff p$
Tautology	$A \cup A^c = U$	$p \vee (\neg p) \iff T$
Contradiction	$A \cap A^c = \emptyset$	$p \wedge (\neg p) \iff F$
Equivalence	$A = B \iff A \subseteq B$ and $B \subseteq A$	$p \leftrightarrow q \iff (p \to q) \wedge (q \to p)$

Table A1: Set identities.

More generally, the highly useful De Morgan's laws say that for *any* index set I,

$$(\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c \quad \text{and} \quad (\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c.$$

References

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