15 Sequences

15.1 Introduction to sequences

Formally an *infinite sequence* is a function $a : \mathbb{N} \to \mathbb{R}$, Informally, a sequence[237] is an ordered list of real numbers:

$$(a_1, a_2, a_3, \ldots) \quad \text{or} \quad (a_k)_{k=1}^{\infty}.$$

We sometimes even just write (a_n), if it clear from the context that this is representing a sequence. Some remarks:

- The number a_k (formally, the image of k under the map a), is called the kth *term* of the sequence. Notice that we write a_k rather than $a(k)$; this is a tradition, but not a requirement.

- Spivak writes $\{a_1, a_2, a_3, \ldots\}$ or $\{a_k\}_{k=1}^{\infty}$. I much prefer "$(\cdots)$" to "$\{\cdots\}$"; because we use "$\{\cdots\}$" for a *set* of elements, this notation might incorrectly convey (at a subconscious level) that the order of the elements in a sequence doesn't matter.

- A sequence doesn't necessarily have to start at element a_1; it will be useful to allow sequences of the form $(a_k, a_{k+1}, a_{k+2}, \ldots)$ (denoted also $(a_j)_{j=k}^{\infty}$) for arbitrary integers k. In particular we will very frequently work with sequences of the form $(a_0, a_1, a_2, \ldots)$.

We give some examples, mostly to indicate different ways that sequences might be presented:

- $a : \mathbb{N} \to \mathbb{R}$, $a(k) = 2 + (-1)^k$.

- $a_n = n^2 + 1$, $n = 0, 1, 2, \ldots$.

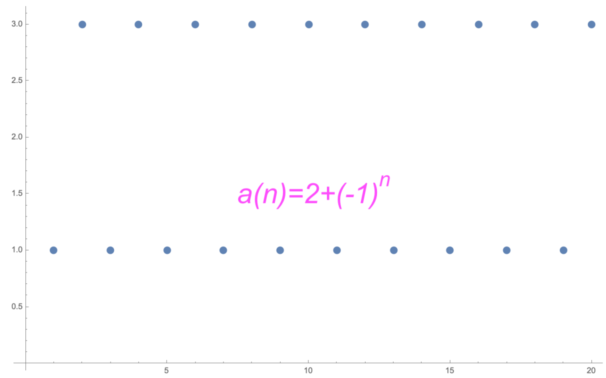
- $(2, 3, 5, 7, 11, 13, 17, \ldots)$. (This is a very typical way to present a sequence — list only a few terms, and let the pattern speak for itself. But it should be used with caution. Is the pattern *really* obvious? In this case, I think that the answer is *no*, since the sequence I'm thinking of dies *not* have 19 as its next element.)

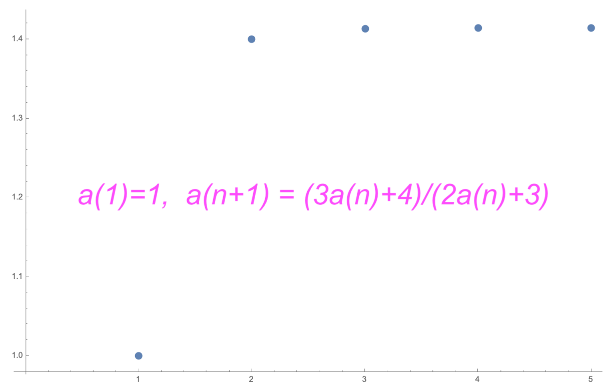
- $a_n = \sum_{k=1}^{n} x^k/k!$. (Notice that this is a *family* of sequences, one for each $x \in \mathbb{R}$.)

- $a_1 = 1$, and, for $n \geq 1$, $a_{n+1} = \frac{3a_n+4}{2a_n+3}$. (This is a *recursive* (or *recursively defined*) sequence.)

A sequence can be graphically represented: here are some examples:

[237] From here on we will typically say "sequence" rather than "infinite sequence".





As illustrated by the last two examples, a sequence can exhibit very different behaviors as the terms get larger.

15.2 Convergence

Definition of a sequence converging A sequence $(a_n)_{n=1}^{\infty}$ *converges to a limit* L *as* n *approaches infinity*, written

- $(a_n) \to L$ as $n \to \infty$
- $a_n \to L$ as $n \to \infty$
- $(a_1, a_2, \ldots) \to L$ as $n \to \infty$
- $\lim_{n\to\infty} a_n = L$,

if for all $\varepsilon > 0$ there exists n_0 such that

$$n > n_0 \quad \text{implies} |a_n - L| < \varepsilon.$$

If a sequence converges to a limit L as n approaches infinity then it is said to be a *convergent* sequence. If a sequence does not converge to a limit, then it is said to *diverge*, or be a *divergent* sequence.

Definition of a sequence converging to ∞ A divergent sequence $(a_n)_{n=1}^{\infty}$ *converges to* ∞ *as* n *approaches infinity*, written

- $(a_n) \to \infty$ as $n \to \infty$
- $a_n \to \infty$ as $n \to \infty$
- $(a_1, a_2, \ldots) \to \infty$ as $n \to \infty$
- $\lim_{n\to\infty} a_n = \infty$,

if for all $N > 0$ there exists n_0 such that

$$n > n_0 \quad \text{implies} \quad a_n > N.$$

The definition of a sequence converging to $-\infty$ is analogous.

A note on notation: any way of notating the concept that the function f approaches a limit near a, it is necessary to include some reference to the parameter a; but when notating the concept that the sequence (a_n) converges to a limit as n approaches infinity, it is usually unnecessary to include reference to the fact that n is approaching infinity; usually it will be perfectly clear from the context that the only place that n can go is to infinity. So we will often write simply:

- $(a_n) \to \infty$
- $a_n \to \infty$,
- $(a_1, a_2, \ldots) \to \infty$.

We illustrate the concept of convergence with three examples:

- $\left(\frac{n-1}{n+1}\right)_{n\geq 1}$. Evidently this converges to 1 as $n \to \infty$. To prove this formally, note that for each $\varepsilon > 0$ we require an n_0 such that $n > n_0$ implies
$$\left|\frac{n-1}{n+1} - 1\right| < \varepsilon.$$
This is equivalent to
$$\left|\frac{(n+1)-2}{n+1} - 1\right| < \varepsilon$$
or
$$\frac{2}{n+1} < \varepsilon$$
or
$$n > \frac{2}{\varepsilon} - 1.$$
So taking $n_0 = (2/\varepsilon) - 1$ we get that the sequence converges to 1 as $n \to \infty$.

- $(n^2 + 1)_{n=1}^{\infty}$. This evidently diverges, and tends to ∞. To verify this formally, we need to show that for each N, there is n_0 such that $n > n_0$ implies $n^2 + 1 > N$. If $N \leq 1$, simply take $n_0 = 1$; if $N > 1$ take n_0 to be anything greater than $\sqrt{N-1}$.

- $(2+(-1)^k)_{k\geq 1}$. This evidently diverges. To see this formally, suppose that it converges to a limit L. Whatever L is, it must be at distance at least 1 from at least one of $1, 3$. Suppose it is distance at least 1 from 3. Take $\varepsilon = 1/2$. By the assumption that the limit is L, there is n_0 such that for all $n > n_0$, $2+(-1)^n$ is within $1/2$ of L. This implies that for all $n > n_0$, $2+(-1)^n$ cannot take the value 3 (since 3 is not within $1/2$ of L, by assumption). But this is a contradiction, since $2+(-1)^n$ takes the value 3 for all even n. We get a similar contradiction under the assumption that the distance from L to 1 is at least 1. So the assumption that the sequence converges to a limit L is untenable, regardless of the choice of L, and $(2+(-1)^k)_{k\geq 1}$ diverges. Note that because $2+(-1)^k$ always lies between 1 and 3, we easily see that $(2+(-1)^k)_{k\geq 1}$ does not converge to either of $\infty, -\infty$ either.

Just as with limits of functions, it is quite annoying to compute limits of sequences directly from the definition. Fortunately, just as in the functions case, there are some basic facts about limits of sequences that allow for relatively straightforward calculation of limits without employing ε-n_0 formalism. These facts mostly mirror those concerning limits of functions.

Theorem 15.1. *We have the following facts.*

- *If a sequence (a'_n) is obtained from the sequence (a_n) by changing finitely many of the a_n (where "changing" includes "making undefined"), then the behavior of both sequence as $n \to \infty$ is the same.*

- *For any natural number k, the sequences (a_{n+k}) and (a_n) have the same limiting behavior as $n \to \infty$.*

- *If (a_n) converges to a limit (including possibly $\pm\infty$), then that limit is unique.*

- *If $(a_n) \to L_a$ and $(b_n) \to L_b$ ($L_a, L_b \in \mathbb{R}$) then*
  - *$(ca_n + db_n) \to cL_a + dL_b$ for any real constants c, d, and*
  - *$(a_n b_n) \to L_a L_b$.*
  - *Moreover, if there is some n_0 such that for all $n > n_0$ we have $a_n = b_n$ then $L_a = L_b$.*

- *If $(a_n) \to L_a$ and $(b_n) \to \infty$ (respectively, $-\infty$) then $(a_n + b_n) \to \infty$ (respectively, $-\infty$).*

- *If (a_n) converges to a limit $L \neq 0$ then*
  - *there is some n_0 such that for all $n > n_0$, a_n is within $L/2$ of L (and so in particular is either always positive or always negative, depending on whether L is positive or negative), and*
  - *$(1/a_n)$ converges to $1/L$.*

- *If (a_n) converges to ∞ (respectively, $-\infty$), then*
  - *for every positive constant C (respectively, negative constant C) there is some n_0 such that for all $n > n_0$, $a_n > C$ (respectively, $a_n < C$) (and so in particular a_n is eventually either always positive or always negative, depending on whether the limit is $+\infty$ or $-\infty$), and*
  - *$(1/a_n)$ converges to 0.*
- *$(1) \to 1$ and $(n) \to \infty$.*
- *If $p(n)$ is polynomial of degree r, with the coefficient n^r being c_r, and $q(n)$ a polynomial of degree s, with the coefficient of n^s being 1, then*

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} c_r & \text{if } r = s \\ 0 & \text{if } s > r \\ +\infty & \text{if } r > s,\ c_r > 0 \\ -\infty & \text{if } r > s,\ c_r < 0 \end{cases}$$

Proof: We leave all of these as exercises! The proofs here are very similar to the proofs of similar statements for limits of functions, and this theorem is a good exercise in reviewing those proofs. □

Armed with this theorem we can easily say, for example, that

$$\left(\frac{2n^4 - n + 1}{n^3 + 1}\right) \to \infty, \quad \left(\frac{2n^4 - n + 1}{n^4 + 1}\right) \to 2 \quad \text{and} \quad \left(\frac{2n^4 - n + 1}{n^5 + 1}\right) \to 0.$$

15.3 Sequences and functions

There is a natural (collection of) connections between limits of functions and limits of sequences. All three of the following facts are left as (easy) exercises:

1. Given a function $f : [1, \infty) \to \mathbb{R}$ with $\lim_{x\to\infty} f(x) = L$ (or ∞, or $-\infty$), define a_n by $a_n = f(n)$. Then $(a_n) \to \ell$ (or ∞, or $-\infty$).

2. The converse of point 1 above is *not* true: if $(a_n) \to \ell$ (or ∞, or $-\infty$) and $f : [1, \infty) \to \mathbb{R}$ satisfies $f(n) = a_n$ for all n, it is not necessarily the case that $\lim_{x\to\infty} f(x) = \ell$ (or ∞, or $-\infty$).

3. However, point 1 above has a partial converse: if $(a_n) \to \ell$ (or ∞, or $-\infty$) and $f : [1, \infty) \to \mathbb{R}$ satisfies $f(n) = a_n$ for all n, and furthermore $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} f(x) = \ell$ (or ∞, or $-\infty$). (And note that there is always such an f. For example, define $f : [1, \infty) \to \mathbb{R}$ by $f(n) = a_n$ for all $n \in \mathbb{N}$, and then extend f to all of $[1, \infty)$ by linear interpolation (for $x \in (n, n + 1)$, $f(x) = (x - n)f(n + 1) + (n + 1 - x)f(n)$).

For example, consider $\lim_{n\to\infty} a^n$.

- For $a > 0$, we have $a^n = e^{n \log a}$.
  - For $a > 1$ we have $\log a > 0$ and so $\lim_{x\to\infty} e^{x \log a} = \infty$. By point 1 above $\lim_{n\to\infty} a^n = \infty$.
  - For $a < 1$ we have $\log a < 0$ and so $\lim_{x\to\infty} e^{x \log a} = 0$. By point 1 above, $\lim_{n\to\infty} a^n = 0$.
  - Rather trivially, for $a = 1$ we have $\lim_{n\to\infty} a^n = 1$.
- For $a < 0$, we write $a^n = (-1)^n(-a)^n$.
  - For $a > -1$ we have $\lim_{n\to\infty}(-a)^n = 0$ (from earlier), and it is an easy exercise that this implies that $\lim_{n\to\infty} a^n = \lim_{n\to\infty}(-1)^n(-a)^n = 0$.
  - for $a \leq -1$ it is an easy exercise to directly verify that $\lim_{n\to\infty} a^n$ does not exist.

In summary

$$\lim_{n\to\infty} a^n = \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } -1 < a < 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases}$$

The most important connection between limits of sequences and limits of functions is conveyed in the following result:

Theorem 15.2. *Suppose that f is continuous at c and that $(a_n) \to c$. Then $\lim_{n\to\infty} f(a_n) = f(c)$*[238]*.*

Conversely, suppose f is defined at and near c, and that $\lim_{n\to\infty} f(a_n) = f(c)$ for all sequences (a_n) that tend to c. Then f is continuous at c.

Before proving this, we give some examples.

Example 1 For the first, consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = (3a_n + 4)/(2a_n + 3)$. A little computation shows that it is highly plausible that this sequence converges to the limit $\sqrt{2}$. We are not yet in a position to prove this. But, *suppose* we know that (a_n) converges to some limit, say c. We can use Theorem 15.2 to prove that the limit must be $\sqrt{2}$. Indeed, consider the function

$$f(x) = \frac{3x + 4}{2x + 3},$$

[238] This expression may not make sense — not all the a_n may be in the domain of f. However, since f is continuous at c, its domain includes $(c - \Delta, c + \Delta)$ for some $\Delta > 0$; and by definition of limit, there is some n_0 such that for all $n > n_0$, $a_n \in (c - \Delta, c + \Delta)$. So *eventually* the sequence $(f(a_n))_{n=1}^{\infty}$ makes sense. And, as we have seen, this is all that is necessary for the expression $\lim_{n\to\infty} f(a_n)$ to make sense.

which is continuous on all its domain ($\mathbb{R} \setminus \{-3/2\}$), and in particular is continuous at c ($a_n \geq 0$ for all n, so $c \geq 0$). From $(a_n) \to c$ we conclude $(f(a_n)) \to f(c)$. But $f(a_n) = a_{n+1}$, and $(a_{n+1}) \to c$. We conclude that $f(c) = c$, or

$$\frac{3c+4}{2c+3} = c.$$

After some easy algebra, we get that the only non-negative solution to this equation is $c = \sqrt{2}$. We conclude that *if* (a_n) converges, then it must converge to $\sqrt{2}$.

Note that the *if* is important here. Consider the recursively defined sequence $a_1 = 2$ and $a_{n+1} = a_n^2$ for $n \geq 1$. If the limit exists and equals c (clearly positive), then by the continuity of $f(x) = x^2$ at c we get by the same argument as above that $c = c^2$ so $c = 1$. But the limit is clearly not 1; the sequence diverges.

Example 2 As a second example, consider $\lim_{n\to\infty} \sqrt{n + a\sqrt{n}} - \sqrt{n + b\sqrt{n}}$. We have

$$\begin{aligned}
\sqrt{n+a\sqrt{n}} - \sqrt{n+b\sqrt{n}} &= \sqrt{n+a\sqrt{n}} - \sqrt{n+b\sqrt{n}} \left(\frac{\sqrt{n+a\sqrt{n}} + \sqrt{n+b\sqrt{n}}}{\sqrt{n+a\sqrt{n}} + \sqrt{n+b\sqrt{n}}} \right) \\
&= \frac{(a-b)\sqrt{n}}{\sqrt{n+a\sqrt{n}} + \sqrt{n+b\sqrt{n}}} \\
&= \frac{(a-b)}{\sqrt{1+\frac{a}{\sqrt{n}}} + \sqrt{1+\frac{b}{\sqrt{n}}}}.
\end{aligned}$$

Now the function

$$f(x) = \frac{a-b}{\sqrt{1+a\sqrt{x}} + \sqrt{1+b\sqrt{x}}}$$

is continuous at 0, with $f(0) = (a-b)/2$, so from Theorem 15.2 and the fact that $(1/n) \to 0$ we conclude that $f(1/n) \to (a-b)/2$, and so

$$\lim_{n\to\infty} \sqrt{n+a\sqrt{n}} - \sqrt{n+b\sqrt{n}} = \frac{a-b}{2}.$$

Example 3 Fix $a > 0$. What is $\lim_{n\to\infty} a^{1/n}$? Write $a^{1/n}$ as $e^{(\log a)/n}$. We have $(\log a)/n \to 0$ as $n \to \infty$, and the function $f(x) = e^x$ is continuous at 0, so by Theorem 15.2 we get $\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} f((\log a)/n) = f(0) = 1$.

Proof (of Theorem 15.2): First suppose that f is continuous at c and that $(a_n) \to c$. Fix $\varepsilon > 0$. There is $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Also, there is n_0 such that $n > n_0$ implies $|a_n - c| < \delta$, so $|f(a_n) - f(c)| < \varepsilon$. Since ε was arbitrary this shows that $(f(a_n)) \to f(c)$.

For other direction, suppose $\lim_{n\to\infty} f(a_n) = f(c)$ for all sequences (a_n) that tend to c, but that $\lim_{x\to c} f(x) \neq f(c)$. So there is an $\varepsilon > 0$, such that for all $\delta > 0$, there is x_δ with

$|x_\delta - c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon$. Applying this with $\delta = 1/n$ for each $n \in \mathbb{N}$, we get a sequence (x_n) with $|x_n - c| < 1/n$ but $|f(x_n) - f(c)| \geq \varepsilon$, so $(f(x_n)) \not\to f(c)$. But evidently $(x_n) \to c$, contradicting our hypotheses. $\square$

There is a slight modification of this result, that has an almost identical proof:

> Suppose f is defined near c (but not necessarily *at* c) and that $\lim_{x\to c} f(x) = \ell$. If $(a_n) \to c$, and for all large enough n we have $a_n \neq c$, then $\lim_{n\to\infty} f(a_n) = \ell$.
>
> Conversely if f is defined near (but not necessarily at) c, and $\lim_{n\to\infty} f(a_n) = \ell$ for all sequences (a_n) that tend to c and that eventually (for all sufficiently large n) avoid c. Then $\lim_{x\to c} f(x) = \ell$.

We will have no need to use this strengthening, so will say no more about it.

One more useful result concerning sequences and convergence is a very natural "squeeze theorem".

Theorem 15.3. *Let $(a_n), (b_n)$ and (c_n) be sequences with $(a_n), (c_n) \to L$. If eventually (for all $n > n_0$, for some finite n_0) we have $a_n \leq b_n \leq c_n$, then $(b_n) \to L$ also.*

Proof: Fix $\varepsilon > 0$. There is n_1, n_2 such that $n > n_1$ implies $a_n \in (L - \varepsilon, L + \varepsilon)$, and $n > n_2$ implies $c_n \in (L - \varepsilon, L + \varepsilon)$. For $n > \max\{n_0, n_1, n_2\}$ (n_0 as in the statement of the theorem), we have

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

so $b_n \in (L - \varepsilon, L + \varepsilon)$. $\square$

Consider, for example,

$$\lim_{n\to\infty} \left(\frac{2n^2 - 1}{3n^2 + n + 2}\right)^{\frac{1}{n}}.$$

We have $(2n^2 - 1)/(3n^2 + n + 2) \to 2/3$ as $n \to \infty$, so for all sufficiently large n

$$0.6^{\frac{1}{n}} \leq \left(\frac{2n^2 - 1}{3n^2 + n + 2}\right)^{\frac{1}{n}} \leq 0.7^{\frac{1}{n}}.$$

Since, as we have seen previously, both $0.6^{\frac{1}{n}}, 0.7^{\frac{1}{n}} \to 1$ as $n \to \infty$, the squeeze theorem allows us to conclude

$$\lim_{n\to\infty} \left(\frac{2n^2 - 1}{3n^2 + n + 2}\right)^{\frac{1}{n}} = 1.$$

There is an "infinite" variant of the squeeze theorem, whose simple proof we omit.

> If $(a_n) \to \infty$ and (b_n) is such that eventually $b_n \geq a_n$, then $(b_n) \to \infty$ also.

15.4 Monotonicity, subsequences and Bolzano-Weierstrass

Definition of a sequence increasing/decreasing A sequence (a_n) is said to be *increasing* (a.k.a. *strictly increasing*) if $a_n > a_m$ whenever $n > m$. It is said to be *weakly increasing* (a.k.a. *non-decreasing*) if $a_n \geq a_m$ whenever $n > m$. The analogous definitions of *decreasing* (a.k.a. *strictly decreasing*) and *weakly decreasing* (a.k.a. *non-increasing*) are omitted. The sequence is said to be *monotone* (a.k.a. *strictly monotone*) if it is either increasing or decreasing, and *weakly monotone* if it is either non-decreasing or non-increasing.

Definition of a sequence being bounded A sequence (a_n) is said to be *bounded above* if there is M such that $a_n \leq M$ for all n, and *bounded below* if there is m such that $m \leq a_n$ for all n. It is said to be bounded if it is both bounded above and bounded below.

Note that

- (a_n) is bounded if and only if there is M such that for all n, $|a_n| \leq M$, and
- If there is a number M' such that for all $n > n_0$ we have $a_n < M'$, then (a_n) is bounded, for example by $\max\{a_1, \ldots, a_n, M'\}$. So, as with converging to a limit, the property of being bounded is one that is not compromised by changing a sequence at finitely many values.

If a sequence (a_n) is bounded above, then $\{a_n : n \in \mathbb{N}\}$ is non-empty and bounded above, so $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ exists. It is certainly not necessarily the case, though, that (a_n) converges under these circumstances, nor the limit, if it exists, has to be α. If, however, the sequence is also non-decreasing, the story is different.

Lemma 15.4. *If (a_n) is non-decreasing and bounded above then $(a_n) \to \alpha := \sup\{a_n : n \in \mathbb{N}\}$.*

Proof: Let $\varepsilon > 0$ be given. There is some n_0 with $a_{n_0} \in (\alpha - \varepsilon, \alpha]$ (otherwise, $\alpha - \varepsilon$ would be an upper bound for (a_n), contradicting that α is the least upper bound. Since (a_n) is non-decreasing we have that $a_n \in (\alpha - \varepsilon, \alpha]$ for all $n > n_0$, so $|a_n - \alpha| < \varepsilon$ for all such n. □

The analogous result, that a non-increasing sequence that is bounded below tends to a limit, and that that limit is $\inf\{a_n : n \in \mathbb{N}\}$, is proven almost identically.

As an example, consider the recursively defined sequence $a_1 = 1$, $a_{n+1} = (3a_n+4)/(2a_n+3)$ for $n \geq 1$. We showed previously that *if* this sequence converges to a limit, that limit must be $\sqrt{2}$. We now show that it does converge to a limit, by showing that it is non-decreasing and bounded above.

We first note that obviously $a_n > 0$ for all n. We have

$$\begin{aligned} a_{n+1} \geq a_n \quad &\text{if and only if} \quad \frac{3a_n + 4}{2a_n + 3} \geq a_n \\ &\text{if and only if} \quad 4 \geq 2a_n^2 \\ &\text{if and only if} \quad a_n \leq \sqrt{2}. \end{aligned}$$

We now show by induction on n that $a_n \leq \sqrt{2}$ for all n; as well as this showing that (a_n) is non-decreasing, it also shows that (a_n) is bounded above, so by the lemma converges.

The base case of the induction is trivial. For the induction step, we assume $a_n \leq \sqrt{2}$ for some $n \geq 1$. We have

$$\begin{aligned} a_{n+1} \leq \sqrt{2} \quad &\text{if and only if} \quad \frac{3a_n + 4}{2a_n + 3} \leq \sqrt{2} \\ &\text{if and only if} \quad 3a_n + 4 \leq \sqrt{2}(2a_n + 3) \\ &\text{if and only if} \quad 9a_n^2 + 24a_n + 16 \leq 8a_n^2 + 24a_n + 18 \\ &\text{if and only if} \quad a_n \leq \sqrt{2}. \end{aligned}$$

This completes the induction, and the verification that $(a_n) \to \sqrt{2}$. Notice that $(a_1, a_2, a_3, \ldots)$ is a list of every-better rational approximations to $\sqrt{2}$.

In general, determining whether a sequence is bounded above or not is not easy! Consider, for example:

- (a_n) where $a_n = 1 + 1/2 + 1/3 + \cdots + 1/n$,
- (b_n) where $b_n = \sum_{p \leq n,\ p \text{ a prime number}} 1/p$, and
- (c_n) where $c_n = \sum_{k \leq n,\ k \text{ has no 7 in its decimal representation}} 1/k$.

We will shortly develop techniques to a test sequences of this form — sequences who generic terms are sums of other sequences — for boundedness.

We now turn to considering *subsequences*. Informally, a subsequence of a sequence

$$(a_1, a_2, a_3, \ldots)$$

is a sequence of the form

$$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$$

with $n_1 < n_2 < n_3 \cdots$. In other words, it is a sequence obtained from another sequence by extracting an infinite subset of the elements of the original sequence, *keeping the elements in the same order as they were in the original sequence.*

Formally, a subsequence is a restriction of a sequence $a : \mathbb{N} \to \mathbb{R}$ to an infinite subset S of $\mathbb{N}$, that is, a function $a|_S : S \to \mathbb{N}$ defined by $a|_S(n) = a(n)$ for $n \in S$.

Here is the fundamental lemma concerning subsequences of a sequence.

Lemma 15.5. *Every sequence has a subsequence which is either non-decreasing or non-increasing. In fact, every sequence has a subsequence which is either weakly increasing or strictly decreasing. Also, every sequence has a subsequence which is either strictly increasing or weakly decreasing.*

Proof: Call a term a_n in a sequence a *horizon point* if $a_n > a_m$ for all $m > n$. If a sequence has infinitely many horizon points, say $a_{n_1}, a_{n_2}, \ldots$, then we get a strictly decreasing subsequence. If there are only finitely many horizon points, then pick a_{n_1} after the last horizon point. Since a_{n_1} is not a horizon point, there is $n_2 > n_1$ with $a_{n_2} \geq a_{n_1}$. Since a_{n_2} is not a horizon point, there is $n_3 > n_2$ with $a_{n_3} \geq a_{n_2}$. Repeating, we get a weakly increasing subsequence.

This shows that every sequence has a subsequence which is either weakly increasing or strictly decreasing. Applying this result to the sequence $(-a_n)$ shows that (a_n) also has a subsequence which is either strictly increasing or weakly decreasing. □

Combining Lemmas 15.4 and 15.5 we get the following, one of the cornerstone theorems of analysis.

Theorem 15.6. *(Bolzano-Weierstrass) If (a_n) is bounded then it has a convergent subsequence.*

Here is a consequence of the Bolzano-Weierstrass theorem. Suppose (a_n) is a bounded sequence, bounded, say, by M (so $-M \leq a_n \leq M$ for all n). Let S be the set of all numbers s such that (a_n) has a subsequence which converges to s. We have that S is non-empty (by Bolzano-Weierstrass). Also, by the squeeze theorem every element in S lies between $-M$ and M. So S is bounded, both from above and from below. By the completeness axiom, then, S has both a supremum and an infimum.

Defintion of $\limsup$ **and** $\liminf$ With the notation as above, the *limit superior* of the sequence (a_n), or *lim sup*, denoted $\limsup a_n$, is the supremum of S, and the *limit inferior*, or *lim inf*, denoted $\liminf a_n$, is the infimum of S.

Think of $\limsup a_n$ as the "largest" of all subsequential limits of (a_n), and $\liminf a_n$ as the "smallest" of all subsequential limits. The quotes around "largest" and "smallest" are there since, as usual when working with infima and suprema, there may not actually be subsequences that converge to the lim sup or lim inf. But in fact they are unnecessary in this case.

Lemma 15.7. *Suppose that (a_n) is a bounded sequence, with $\limsup a_n = \alpha$ and $\liminf a_n = \beta$. Then (a_n) has a subsequence that converges to α, and one that converges to β.*

Proof: We'll just show that there is a sequence that converges to α; the proof of the existence of a sequence converging to β is similar.

If $\alpha \in S$, we are immediately done. If not, for each $n \in \mathbb{N}$ there is $\alpha_n \in S$ with $\alpha - 1/n < \alpha_n < \alpha$ (since $\alpha = \sup S$), and there is a subsequence that converges to α_n.

Build a new subsequence as follows: chose a_{n_1} to be any term of the subsequence converging to α_1, that is distance less than 1 from α_1; chose a_{n_2} to be any term of the subsequence converging to α_2, that is distance less than $1/2$ from α_2; and in general chose a_{n_k} to be any term of the subsequence converging to α_k, that is distance less than $1/k$ from α_k.

Notice that a_{n_1} is distance less than $1+1=2$ from α; a_{n_1} is distance less than $1/2+1/2=1$ from α; and in general a_{n_k} is distance less than $1/k+1/k=2/k$ from α. Since $2/k \to 0$ as $n \to \infty$, it follows that for any $\varepsilon > 0$ the terms of the sequence $(a_{n_1}, a_{n_2}, \ldots)$ eventually are all within ε of α. □

So:

> The lim sup of a bounded sequence is the *largest* subsequential limit, and the lim inf is the *smallest* subsequential limit.

A bounded sequence may not have a limit, but it always has a lim inf and a lim sup, and it is for this reason that these parameters are introduced. As an example, the sequence whose nth term a_n is 0 if n is odd, and $1-(1/n)$ if n is even, has no limit, but it has $\liminf a_n = 0$ and $\limsup a_n = 1$.

The notions of lim sup and lim inf can be thought of as generalizations of the notion of the limit of a sequence, because if a sequence (a_n) has a limit, then it is fairly easy to prove (exercise!) that $\lim a_n = \limsup a_n = \liminf a_n$ (and conversely, if (a_n) is a sequence with $\limsup a_n = \liminf a_n$ then the sequence converges to the common value).

lim sup and lim inf can also be thought of as capturing the "eventual" behavior of a sequence. Suppose (a_n) is a sequence with $\limsup a_n = \alpha$. For each $\varepsilon > 0$, it must be the case that only finitely many terms of the sequence are larger than $\alpha + \varepsilon$ (if not, there would be a subsequence consisting only of terms larger than $\alpha + \varepsilon$, and by the Bolzano-Weierstrass theorem this subsequence would have a subsequence converging to a limit that lies at or above $\alpha + \varepsilon$, contradicting that α is the lim sup). On the other hand, for each $\varepsilon > 0$, it must be the case that infinitely many terms of the sequence are larger than $\alpha - \varepsilon$ (by definition of α there must be a subsequence converging to some value between $\alpha - \varepsilon/2$ and α, and that subsequence eventually always has terms greater than $\alpha - \varepsilon$). This leads to an alternate characterization of lim sup and lim inf (we skip the nitty gritty details of verifying this):

> If (a_n) is a bounded sequence, then $\limsup a_n$ is the unique real number α such that for each $\varepsilon > 0$ only finitely many terms of the sequence are larger than $\alpha + \varepsilon$, while infinitely many terms of the sequence are larger than $\alpha - \varepsilon$. Also $\liminf a_n$ is the unique real number β such that for each $\varepsilon > 0$ only finitely many terms of the sequence are smaller than $\beta - \varepsilon$, while infinitely many terms of the sequence are smaller than $\beta + \varepsilon$.

There are other characterizations of lim sup and lim inf, and extensions to unbounded sequences, which we do not address.

We end our discussion of convergence of sequences by introducing one last test for convergence.

Definition of a Cauchy sequence A sequence (a_n) is *Cauchy*, or *a Cauchy sequence*, if for all $\varepsilon > 0$ there is n_0 such that $n, m > n_0$ implies $|a_n - a_m| < \varepsilon$.

In other words, a sequence is Cauchy not necessarily if the terms eventually get close to a particular limit, but if they eventually get close to *one another*.

For example, every sequence (a_n) that converges to a limit, is Cauchy. Indeed, suppose that the limit is L. Fix $\varepsilon > 0$. There is n_0 such that $n, m > n_0$ implies both $|a_n - L| < \varepsilon/2$ and $|a_m - L| = L - a_m| < \varepsilon/2$. But then,

$$|a_n - a_m| = |a_n - L + L - a_m| \le |a_n - L| + |L - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

In fact, convergent sequences are the *only* Cauchy sequences:

Lemma 15.8. *If (a_n) is a Cauchy sequence, then (a_n) converges.*

Proof: Let (a_n) be a Cauchy sequence. The proof that (a_n) converges goes in three steps.

- (a_n) **is bounded**: There is n_0 such that $n, m > n_0$ implies $|a_n - a_m| < 1$. In particular, for every $m > n_0$, $|a_{n_0+1} - a_m| < 1$, so $a_m \in (a_{n_0+1} - 1, a_{n_0+1} + 1)$. So (a_n) is bounded above by $\max\{a_1, \ldots, a_{n_0}, a_{n_0+1} + 1\}$, and below by $\min\{a_1, \ldots, a_{n_0}, a_{n_0+1} - 1\}$.

- (a_n) **has a convergent subsequence**: Directly from the Bolzano-Weierstrass theorem, there is a subsequence of (a_n), $(a_{n_1}, a_{n_2}, \ldots)$ say, that converges to a limit, L say.

- (a_n) **converges to** L: Suppose not. Then there is $\varepsilon > 0$ for which it is not the case that eventually all terms of the sequence are within ε of L. In other words, there is a subsequence $(a_{n'_1}, a_{n'_2}, \ldots)$ with $|a_{n'_k} - L| \ge \varepsilon$. Now pick any n_0. There is $n'_k > n_0$; there is also $n_k > n_0$ such that $|a_{n_k} - L| < \varepsilon/10$. It follows that $|a_{n_k} - a_{n'_k}| > \varepsilon/2$,[239] and so it is not possible to find an n_0 such that for all $n, m > n_0$, we have $|a_n - a_m| < \varepsilon/2$. This contradicts that (a_n) is Cauchy.

□

Showing that a sequence is Cauchy allows us to show that it is convergent, without

[239] Draw a picture!

actually finding the limit. Consider, for example, $a_n = \sum_{k=1}^{n} 1/k^2$. We have, for $n > m$,

$$\begin{aligned}
|a_n - a_m| &= \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots + \frac{1}{n^2} \\
&\leq \frac{1}{(m+1)^2 - (m+1)} + \frac{1}{(m+2)^2 - (m+2)} + \cdots + \frac{1}{n^2 - n} \\
&= \left(\frac{1}{m} - \frac{1}{m+1}\right) + \left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= \frac{1}{m} - \frac{1}{n} \\
&\leq \frac{1}{m} \\
&< \frac{1}{n_0}.
\end{aligned}$$

So, given $\varepsilon > 0$, if we choose n_0 such that $1/n_0 \leq \varepsilon$, then for all $n, m > n_0$ we have $|a_n - a_m| < \varepsilon$, showing that (a_n) is Cauchy, and so converges to a limit.

Notice that we were able here to establish that (a_n) converges, without actually identifying the limit. This illustrates the value of the concept of Cauchy sequences.[240]

[240]The problem of evaluating the limit of (a_n) in this case is known as the *Basel problem*, and was famously solved by Euler, who showed the remarkable formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

See e.g. https://en.wikipedia.org/wiki/Basel_problem.