16 Series

16.1 Introduction to series

Informally a *series* or *infinite series* is an expression of the form

$$a_1 + a_2 + a_3 + \dots$$

or

$$\sum_{k=1}^{\infty} a_k.$$

We clearly have to take care with such an expression, as it is far from clear that the operation of adding infinitely many things is well defined. For example, we could argue

$$\sum_{k=1}^{\infty} (-1)^k = 0$$

by writing

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^k &= (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 0 + 0 + 0 + \cdots \\ &= 0; \end{aligned}$$

but we could equally well argue

$$\sum_{k=1}^{\infty} (-1)^k = -1$$

by writing

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^k &= -1 + (1-1) + (1-1) + \cdots \\ &= -1 + 0 + 0 + \cdots \\ &= -1. \end{aligned}$$

We will see more startling paradoxes later.

Formally, given a sequence (a_n), define the *nth partial sum* of (a_n) by

$$s_n = a_1 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

Definition of summability Say that (a_n) is *summable* if (s_n) converges to some limit ℓ.

If (a_n) is summable we write $\sum_{k=1}^{\infty} a_k$ or $a_1 + a_2 + \ldots$ for ℓ. Informally, we say "the series $\sum_{k=1}^{\infty} a_k$ *converges* (to ℓ)"[241], or

$$\sum_{k=1}^{\infty} a_k = \ell.$$

We give some examples here:

- if (a_n) is eventually (for all sufficiently large n) 0, then it is summable.
- $((-1)^n)$ is *not* summable: the sequence of partial sums is

$$(-1, 0, -1, 0, -1, 0 \ldots),$$

  which does *not* converge to a limit.
- $(1/n)$ is *not* summable. The nth partial sum s_n is

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

  There are many ways to see that the sequence (s_n) does not tend to a limit. Perhaps the simplest is to consider the subsequence (s_{2^n}). We have

$$\begin{aligned} s_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^n}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \frac{1}{2^n}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + \frac{n}{2}. \end{aligned}$$

  Since $1 + n/2$ can be made arbitrarily large by choosing n large enough, we see that (s_n) cannot possibly tend to a finite limit.[242]
- Consider the sequence $(r^n)_{n=0}^{\infty}$ with $|r| < 1$. We have

$$s_n = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r} - \frac{r^{n+1}}{1 - r}$$

  For $|r| < 1$, $\lim_{n\to\infty} r^{n+1} = 0$, so $(s_n) \to 1/(1 - r)$ as $n \to \infty$. We conclude that

  for $|r| < 1$, $(r^n)_{n=0}^{\infty}$ is summable, and

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

  This is the incredibly useful *geometric series sum*.

[241] Note that this is quite informal; the expression "$\sum_{k=1}^{\infty} a_k$" is just a single expression, that is not varying, so is not really in any sense converging to anything

[242] The partial sum $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is called the *nth Harmonic number*, usually denoted H_n. It's properties will be explored in a homework problem.

16.2 Tests for summability

We now develop a collection of tests/criteria for summability.

Basic closure properties If (a_n), (b_n) are both summable, then so are

- $(a_n + b_n)$ — with
$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$
- and (ca_n) — with
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

The proofs of these facts are left as exercises.

For all $k \geq 0$, the sequences $(a_1, a_2, a_3, \ldots)$ and $(a_k, a_{k+1}, a_{k+2}, \ldots)$ are either both summable or both not, and if they are both summable, then they have the same sum; from this it follows that if two sequences can be made equal by shifting and changing finitely many terms, then they are either both summable or both not. Again, the proof is left as an exercise.

Cauchy criterion (a_n) is summable if and only if (s_n) is Cauchy. This follows from Lemma 15.8, and from the observation before that lemma that convergent sequences are Cauchy. This means that (a_n) is summable if and only if for all $\varepsilon > 0$ there's n_0 such that $n, m > n_0$ implies $|s_m - s_n| < \varepsilon$, that is (assuming without loss of generality that $m > n$)
$$|a_{n+1} + a_{n+2} + \ldots + a_m| < \varepsilon.$$

The intuition here is that a sum $\sum_{n=1}^{\infty} a_n$ converges if and only if its "tail" $a_n + a_{n+1} + a_{n+2} + \cdots$ can be made arbitrarily small. But this is just an intuition; the tail of an infinite sum is itself a sum of infinitely many things, and so to properly understand it we need the theory that we are in the process of developing. The Cauchy criterion expresses the idea that the tail of the sequence can be made arbitrarily small, while only ever referring to the sum of finitely many terms.

Vanishing condition If (a_n) is summable then, from the Cauchy condition, for all $\varepsilon > 0$ there is n_0 such that $n, m > n_0$ implies $|s_n - s_m| < \varepsilon$. Applying this with $m = n - 1$ we get that for sufficiently large n, $|a_n| < \varepsilon$. It follows that $\lim_{n\to\infty} |a_n| = 0$, so $\lim_{n\to\infty} a_n = 0$. The contrapositive of this is what is usually used:

if $\lim_{n\to\infty} a_n \neq 0$ then (a_n) is *not* summable.

For example, for $|r| \geq 1$ we have $\lim_{n\to\infty} r^n \neq 0$, so in this range (r^n) *not* summable (we have already seem that it is summable, with sum $1/(1 - r)$ for all other r).

Note: the converse of vanishing condition is **not** true — $(a_n) \to 0$ does *not* imply that (a_n) is summable. An example to consider is $(1/n)$.

Boundedness criterion This is more a theoretical than a practical criterion. If $a_n \geq 0$ for all n, then (s_n) is increasing. So in this situation

(a_n) is summable iff (s_n) is bounded above.

As an example, we earlier showed that $\left(\sum_{k=1}^{n} \frac{1}{k^2}\right)$ is bounded above, and so now we know that $(1/n^2)$ is summable, i.e., that the expression $\sum_{n=1}^{\infty} \frac{1}{n^2}$ makes sense (is finite).

Comparison test Suppose $b_n \geq a_n \geq 0$ for all n. If (b_n) is summable, then so is (a_n) — this is because the partial sums of (a_n) are bounded above by $\sum_{n=1}^{\infty} b_n$, so we can apply the boundedness criterion. Contrapositively, if (a_n) is not summable, then neither is (b_n).

We give some examples.

- $(1/n^\alpha)$, $\alpha < 1$. We have $n^\alpha < n$, so $1/n^\alpha > 1/n \geq 0$. By comparison with $(1/n)$, $(1/n^\alpha)$ is not summable.
- $(n^3/3^n)$. We know that $(1/3^n)$ is summable, so we would like to say that $(\frac{n^3}{3^n})$ is too, by comparison. But the inequality goes the wrong way — we have $1/3^n \leq \frac{n^3}{3^n}$. We can, however, compare with $(1/2^n)$. We have $n^3/3^n \leq 1/2^n$ for all large enough n, so by comparison $(\frac{n^3}{3^n})$ is summable.[243]
- $(n^2/(n^3+1))$. This looks a lot like $1/n$, so we suspect that it is not summable. But it is not true that $n^2/(n^3+1) > 1/n$. However, for all large enough n, we have $n^2/(n^3+1) > 1/2n$ (actually for $n > 1$), so $(n^2/(n^3+1))$ is not summable, by comparison with $(1/2n)$.

Limit comparison test As the last few examples show, sometimes the comparison test can be awkward to apply. A much more convenient version is the limit comparison test:

Suppose $a_n, b_n > 0$ and $\lim_{n\to\infty} a_n/b_n = c > 0$. Then (a_n) is summable if and only if (b_n) is.

To prove this, first suppose that (b_n) is summable. There is n_0 such that $n > n_0$ implies $a_n < 2cb_n$. Since (b_n) is summable, so is $(2cb_n)$. We can now conclude that (a_n) is summable, by comparison with $(2cb_n)$. (As usual, we are ignoring finitely many terms of (a_n) that might be larger than their companion terms in $(2cb_n)$).

In the other direction, if (a_n) is summable, then since $b_n/a_n \to 1/c > 0$ we get that (b_n) is summable by the above argument.

We give a few examples:

[243] We don't care what happens for finitely many n; that changes the sum, but not whether the sequence is summable.

- $(n^2/(n^3+1))$. We have
$$\lim_{n\to\infty} \frac{n^2/(n^3+1)}{1/n} = 1,$$
so by limit comparison with $(1/n)$, we get that $(n^2/(n^3+1))$ is not summable (and note that this went more smoothly than the comparison test).
- $(2/\sqrt[3]{n^2+1})$. Since $1/n^{2/3}$ diverges, and
$$\lim_{n\to\infty} \frac{2/\sqrt[3]{n^2+1}}{1/n^{2/3}} = 2,$$
we get that $(2/\sqrt[3]{n^2+1})$ is not summable.

Ratio test This is possibly the most useful criterion for summability. If $a_n > 0$ for all (sufficiently large) n, and
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$
then:

- if $r < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges (that is, (a_n) is summable),
- if $r > 1$, the series does not converge, and
- if $r = 1$, no conclusion can be reached.

Before proving this, we give some examples.

- If $a_n = x^{2n}/(2n)!$ (positive for all n and all $x \in \mathbb{R}$), then $\lim_{n\to\infty} a_n = 0$, and so
$$\sum_{n\geq 0} \frac{x^{2n}}{(2n)!}$$
converges to a limit for all real x. By the same argument, so does
$$\sum_{n\geq 0} \frac{x^n}{n!}.$$
We should strongly suspect that the sum is e^x (and in fact we can easily prove this at this point); we will return to this later. Notice that from the summability of $(x^n/n!)$ and the vanishing criterion, we recover a previous result, that
$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$
for all real x.

- $a_n = n^k/c^n$, $c > 1$, $k > 0$. We apply the ratio test, and see that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^k}{cn^k} = \frac{1}{c} < 1$$

  and so (n^k/c^n) summable, that is,

$$\sum_{n=0}^{\infty} \frac{n^k}{c^n}$$

  is a finite number. Note that this tells us (via the vanishing criterion) that $\lim_{n\to\infty} n^k/c^n = 0$ (something we have seen before).

We now turn to the proof of ratio test. Suppose $r < 1$. Fix s with $r < s < 1$. There is n_0 such that $a_{n+1}/a_n < s$ for all $n > n_0$. We have

$$a_{n+1} < sa_n,$$

$$a_{n+2} < sa_{n+1} < s^2 a_n,$$

and in general

$$a_{n+k} < s^k a_n,$$

so applying at $n = n_0 + 1$ we get

$$a_{(n_0+1)+k} < s^k a_{n_0+1}$$

for all $k \geq 1$. Since $(s^k a_{n_0+1})_{k=1}^{\infty}$ is summable (it is a geometric series), so is $(a_{(n_0+1)+k})_{k=1}^{\infty}$ (by comparison), and so so also is $(a_n)_{n\geq 1}$.

On the other hand, suppose $r > 1$. Fix s with $1 < s < r$. There n_0 such that for all $n > n_0$, $a_{n+1}/a_n > s$ so that (by the same reasoning as above) $a_{(n_0+1)+k} > s^k a_{n_0+1}$, so $\lim_{n\to 0} a_n \neq 0$ and so (a_n) not summable.

Finally, the sequences $(1/n)$ (which is not summable) and $(1/n^2)$ (which is) both have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

which verifies that no conclusion can be reached if $r = 1$.

Integral test Suppose that $f : [1,\infty) \to (0,\infty)$ is non-increasing, and $f(n) = a_n$. Then (a_n) is summable if and only if $\int_1^\infty f$ exists.

As an example, consider the *p-series* $\sum_{n=1}^{\infty} 1/n^p$.

- If $p \leq 0$, the sum diverges, by the vanishing criterion.
- $p > 0$, the sum converges if and only if $\int_1^\infty dx/x^p$ exists, which is the case exactly if $p > 1$.

At $p = 1$ we recover the divergence of the Harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

At $p = 2$ we get that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is some finite number (as we have seen before, via ad-hoc methods); that number happens to be $\pi^2/6$.

Here is a proof of the integral test. Consider the sequence whose nth term is $\int_n^{n+1} f$. We have that $\int_1^{\infty} f$ exists if and only if $(\int_n^{n+1} f)_{n\geq 1}$ is summable.

Because f is decreasing we have

$$a_{n+1} \leq \int_n^{n+1} f \leq a_n.$$

Suppose $(\int_n^{n+1} f)_{n\geq 1}$ is summable. Then the first inequality, together with comparison, says that (a_{n+1}) is summable, and so (a_n) is summable.

Suppose on the other hand that (a_n) is summable. Then the second inequality, together with comparison, says that $(\int_n^{n+1} f)_{n\geq 1}$ is summable.

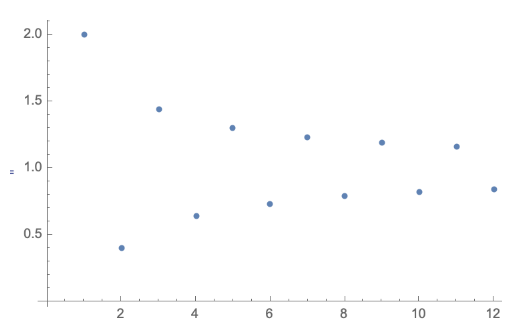
Leibniz' theorem on alternating series Most of the powerful tests for summability presented so far have concerned non-negative sequence. We now present a test which considers sequences that have some negative terms. Suppose (a_n) is non-increasing and tends to 0 (and so necessarily $a_n \geq 0$ for all n). Then Leibniz' theorem is the assertion that the *alternating* series $((-1)^{n-1}a_n)$ is summable, i.e.,

$$a_1 - a_2 + a_3 - a_4 \cdots$$

converges to a (finite) limit.

For example, $(1/n)$ is not summable, but $((-1)^n/n)$ is; and if p_n is the nth prime, then $(1/p_n)$ not summable, but $((-1)^n/p_n)$ is.

To prove Leibniz' theorem, it helps to draw a picture of the sequence of partial sums, which strongly suggests that $s_1 \geq s_3 \geq \cdots \geq \ell$ and $s_2 \leq s_4 \leq s_6 \cdots \leq L$, for some limit L, and with every odd partial sum exceeding every even one:



Inspired by this picture, we prove Leibniz' theorem in small steps:

- First, $s_1 \geq s_3 \geq s_5 \geq \cdots$ (odd partial sums form a non-increasing sequence). Indeed, for $n \geq 1$ we have

$$s_{2n-1} - s_{2n+1} = a_{2n} - a_{2n+1} \geq 0$$

  since (a_n) is non-increasing.

- Next, $s_2 \leq s_4 \leq s_6 \leq \cdots$ (even partial sums form a non-decreasing sequence). Indeed, for $n \geq 2$ we have

$$s_{2n} - s_{2n-2} = a_{2n-1} - a_{2n} \geq 0$$

  again since (a_n) is non-increasing.

- Next, if k is even and ℓ is odd, then $s_k \leq s_\ell$ (all odd partial sums are at least as large as all even partial sums). Indeed, for every n we have $s_{2n-1} - s_{2n} = a_n \geq 0$, so $s_{2n} \leq s_{2n-1}$. So, choosing n large enough that $2n > k$ and $2n - 1 > \ell$, we have from the first two observations that

$$s_k \leq s_{2n} \leq s_{2n-1} \leq s_\ell.$$

- Next, the sequence $(s_{2n})_{n=1}^{\infty}$ converges to a limit, say α. Indeed, it is non-decreasing and bounded above, say by s_1, by previous observations, so converges.

- Next, the sequence $(s_{2n-1})_{n=1}^{\infty}$ converges to a limit, say β. Indeed, it is non-increasing and bounded below, say by s_2, by previous observations, so converges.

- Next, $\alpha = \beta$. Indeed, as previously observed we have $s_{2n-1} - s_{2n} = a_n$. Taking limits of both sides as n goes to infinity, and using the last unused hypothesis (that $(a_n) \to 0$) we get $\beta - \alpha = 0$ so $\alpha = \beta$.

- Finally, letting L be the common value of α, β, we have $(s_n) \to L$. Indeed, fix $\varepsilon > 0$. Because (s_{2n}) increases to limit L there is n_1 such that if $n > n_1$ and n is even, then $|s_n - L| < \varepsilon$, and because (s_{2n-1}) decreases to limit L there is n_2 such that if $n > n_2$ and n is odd, then $|s_n - L| < \varepsilon$. So for $n > \max\{n_1, n_2\}$ we have $|s_n - L| < \varepsilon$.

In fact, this proof gives something more: for each n we have $s_{2n} \leq L \leq s_{2n+1}$, so

$$L - s_{2n} \leq s_{2n+1} - s_{2n} = a_{2n+1},$$

and also $s_{2n+2} \leq L \leq s_{2n+1}$, so

$$s_{2n+1} - L \leq s_{2n+1} - s_{2n+2} = a_{2n+2}.$$

In other words:

Let $S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where $a_n \geq 0$, (a_n) is non-increasing and $(a_n) \to 0$ (so S is finite, by Leibniz' theorem). The sum of the first $2n$ terms (truncating at a subtracted term) underestimates S, but by at most a_{2n+1}, while the sum of the first $2n+1$ terms (truncating at an added term) overestimates S, but by at most a_{2n+2}.

Consider, for example,

$$S = \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{p_n}$$

where p_n is the nth prime number. The number S is finite by Leibniz' theorem. If we want to estimate S to within ± 0.001, we note that the 168th prime number is 997, while the 169th is 1009. So the sum of the first 168 terms of the series is within $1/1009$ of the limit, and moreover this partial sum underestimates the limit. We conclude that

$$\sum_{n=1}^{168} \frac{(-1)^{n-1}}{p_n} \leq S \leq \sum_{n=1}^{168} \frac{(-1)^{n-1}}{p_n} + \frac{1}{1009} \leq \sum_{n=1}^{168} \frac{(-1)^{n-1}}{p_n} + 0.001.$$

A tedious calculation shows

$$\sum_{n=1}^{168} \frac{(-1)^{n-1}}{p_n} = 0.269086\ldots$$

while

$$S = 0.269606351\ldots.$$

16.3 Absolute convergence

Leibniz' theorem allows us to deal with the question of convergence of *alternating* series. For series which have more arbitrary patterns of signs, we need the concept of *absolute convergence*.

Definition of absolute convergence If (a_n) is a sequence of real numbers, we say that (a_n) is *absolutely summable* (or, $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent*) if $(|a_n|)$ is summable (that is, if $\sum_{n=1}^{\infty} |a_n|$ converges). If (a_n) is summable but not absolutely summable we say that is *conditionally summable* (or that $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent*).

For example, $((-1)^{n-1}/n)$ is summable (by Leibniz' theorem) but not absolutely summable (the Harmonic series diverges), and so is an example of a conditionally summable sequence; while if $f : \mathbb{N} \to \{+1, -1\}$ is any function then $(f(n)/2^n)$ is absolutely convergent.

The example of $((-1)^{n-1}/n)$ shows that conditional convergence does not imply convergence. On the other hand, we have the following theorem that makes the notion of absolute convergence a very useful one.

Theorem 16.1. *(Absolute convergence implies convergence) If (a_n) is absolutely summable, it is summable.*

Proof: Fix $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} |a_n|$ converges, we get from the Cauchy criterion that there is n_0 such that

$$|a_{n+1}| + \cdots + |a_m| < \varepsilon$$

for all $m > n > n_0$. But the triangle inequality says

$$|a_{n+1} + \cdots + a_m| < |a_{n+1}| + \cdots + |a_m|$$

and so we have

$$|a_{n+1} + \cdots + a_m| < \varepsilon$$

for all $m > n > n_0$, which says (again by the Cauchy criterion) that $\sum_{n=1}^{\infty} a_n$ converges. □

As a corollary we get that if (a_n) is a non-negative summable sequence, and $f : \mathbb{N} \to \{+1, -1\}$ is *any* function, $(f(n)a_n)$ is summable; this allows us to deal with many "irregularly alternating" series. As a specific example, consider

$$\left(\frac{\sin(n^2+1)}{n\sqrt{n}}\right)_{n=1}^{\infty}.$$

It is very difficult to keep track of the way that the sign of $\sin(n^2 + 1)$ changes as n changes (it does so quite chaotically, starting out

$$1, -1, -1, -1, 1, -1, -1, 1, 1, 1, 1, 1, 1, 1, -1, -1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, 1, -1, 1, 1$$

according to `Mathematica`). But, the sequence is easy seen to be absolutely convergent:

$$0 \leq \left|\frac{\sin(n^2+1)}{n\sqrt{n}}\right| \leq \frac{1}{n\sqrt{n}},$$

so we get absolute convergence by comparison with the p-series $(1/n^{3/2})$ (which converges since $p > 1$); and so we get convergence of the original sequence.

We have previously seen that *finite* addition is commutative: for any three reals a_1, a_2, a_3 we have

$$a_1 + a_2 + a_3 = a_1 + a_3 + a_2 = a_2 + a_1 + a_3 = a_2 + a_3 + a_1 = a_3 + a_1 + a_2 = a_3 + a_2 + a_1$$

and more generally, no matter the order that n reals are arranged, their sum remains unchanged.

What about *infinite* addition? If $(a_n)_{n=1}^{\infty}$ is a summable sequence, does the sum depend on the order in which the a_i are written? Unfortunately[244], the answer is *yes*.

[244] Or maybe **fortunately** — odd results like the one described here make the mathematical landscape richer.

Example: By Leibniz' theorem we have that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\cdots=L \quad (\star)$$

for some finite number L. Whatever L is, by Leibniz' theorem we have $L > 1-(1/2) = 1/2 > 0$ (on truncating a Leibniz alternating series after a subtracted term, the partial sum underestimates the limit).

Exercise: Consider the sum

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}\cdots \quad (\star\star)$$

obtained from the left-hand side of $(\star)$ by rearranging the terms as follows: take the first positive term first, then the first two negative terms, then the next positive term, then the next two negative terms, and so on. By combining the 1 with the $1/2$, the $1/3$ with the $1/6$, the $1/5$ with the $1/10$, and so on (combine two, skip one, repeat), argue that $(\star\star)$ converges to $L/2$ — a different sum to $(\star)$, even though the terms are the same, just written in a different order!

Worse[245] is true:

Theorem 16.2. *If* (a_n) *is* conditionally *summable, then for* any *real number* α *there is a rearrangement*[246] (b_n) *of* (a_n) *with* $\sum_{n=1}^{\infty} b_n = \alpha$*; and there are also rearrangements with* $\sum_{n=1}^{\infty} b_n = \infty$ *and* $\sum_{n=1}^{\infty} b_n = -\infty$*.*

Proof: We proceed in steps. The details are left as exercises.

- Step 1: Let (c_n) be any sequence. Define the *positive part* of (c_n) to be the sequence (c_n^+) given by

$$c_n^+ = \begin{cases} c_n & \text{if } c_n \geq 0 \\ 0 & \text{if } c_n < 0 \end{cases}$$

and define the *negative part* to be (c_n^-) with

$$c_n^- = \begin{cases} c_n & \text{if } c_n \leq 0 \\ 0 & \text{if } c_n > 0. \end{cases}$$

Exercise: Convince yourself that for each n

$$2c_n^+ = c_n + |c_n| \quad \text{and} \quad 2c_n^- = c_n - |c_n|.$$

Deduce that

If $\sum_{n=1}^{\infty} c_n$ is absolutely convergent then both $\sum_{n=1}^{\infty} c_n^+$ and $\sum_{n=1}^{\infty} c_n^-$ converge.

[245] Or better — see above footnote.

[246] so (b_n) has exactly the same terms as (a_n), just perhaps in a different order.

Next convince yourself that for each n

$$|c_n| = c_n^+ - c_n^-. \quad (\star\star\star)$$

Deduce that

> If both $\sum_{n=1}^\infty c_n^+$ and $\sum_{n=1}^\infty c_n^-$ converge then $\sum_{n=1}^\infty c_n$ is absolutely convergent.

These two facts together say that for any arbitrary sequence (c_n)

> $\sum_{n=1}^\infty c_n$ is absolutely convergent if and only if both $\sum_{n=1}^\infty c_n^+$ and $\sum_{n=1}^\infty c_n^-$ converge.

- Step 2: Now let (a_n) be the conditionally summable sequence hypothesized in the theorem. Let p_n be the positive part of (a_n), and let (q_n) be the negative part. Since (a_n) is not *absolutely* summable, by Step 1 of the proof we know that at least one of $\sum_{n=1}^\infty p_n$, $\sum_{n=1}^\infty q_n$ diverges (if the former, then to $+\infty$; if the latter, to $-\infty$).

  Exercise: Suppose $\sum_{n=1}^\infty p_n$ diverges to $+\infty$, but $\sum_{n=1}^\infty q_n$ converges to some fixed number L. Using $a_n = p_n + q_n$ (similar to $(\star\star\star)$ above), argue that $\sum_{n=1}^\infty a_n$ diverges to $+\infty$, a contradiction. (We proved something very like this in class.)

  Similarly if $\sum_{n=1}^\infty p_n$ converges, but $\sum_{n=1}^\infty q_n$ diverges to $-\infty$ then we get the contradiction that $\sum_{n=1}^\infty a_n$ diverges to $-\infty$. So the conclusion of Step 2 is

  > If $\sum_{n=1}^\infty a_n$ is conditionally convergent then both $\sum_{n=1}^\infty p_n$ diverges to $+\infty$ and $\sum_{n=1}^\infty q_n$ diverges to $-\infty$.

- Step 3: (**THE MEAT**) Fix $\alpha > 0$. Construct a rearrangement of (a_n) whose sum converges to α, as follows:[247]
  - Start the rearrangement by taking initial terms from (p_n), until the partial sum of the rearrangement thus far constructed either reaches or exceeds α.
    * **Question**: How do we know that such a point can be reached?
    * **Question**: Suppose it is reached exactly when p_{n_1} is added. By how much at most can the partial sum exceed α at this point?
  - Continue the rearrangement by taking initial terms from (q_n) until the partial sum of the rearrangement thus far constructed either reaches or falls below α.
    * **Question**: How do we know that such a point can be reached?
    * **Question**: Before this point is reached, by how much at most can the partial sums exceed α?

[247] Pictures will be *very* helpful here!

    * **Question**: Suppose this point is reached exactly when q_{n_1} is "added"[248]. By how much at most can the partial sum fall short of α at this point?

  - Continue the rearrangement by going back to where you left off from (p_n), and continuing to take terms from (p_n) until the partial sum of the rearrangement either reaches or exceeds α.

    * **Question**: How do we know that such a point can be reached?
    * **Question**: Before this point is reached, by how much at most can the partial sums fall short of α?
    * **Question**: Suppose this point is reached exactly when p_{n_2} is added. By how much at most can the partial sum exceed α at this point?

  - Continue in this manner, swapping back and forth between (p_n) and (q_n) alternately adding from (p_n) until α is reached or exceeded, then adding from (q_n) until α is reached or fallen short of.

    * **Question**: How do we know that this process can continue indefinitely?
    * **Question**: Suppose that the points where you flip from choosing from one subsequence to the other happen at p_{n_1}, q_{n_1}, p_{n_2}, q_{n_2}, and so on. In terms of these quantities, by how much at most can the partial sums of the rearrangement differ from α?

  Exercise: Use what you know[249] about the sequence $(p_{n_1}, q_{n_1}, p_{n_2}, q_{n_2}, \ldots)$ to conclude that the rearrangement just constructed is summable with sum α.

  - Step 4: Modify the argument to deal with negative α, $\alpha = 0$, $\alpha = \infty$ and $\alpha = -\infty$.

□

The story for *absolutely* convergent sequences vis-à-vis infinite commutativity is completely different. Here's a theorem that that says that if the sequence (a_n) is absolutely convergent, then the order in which the terms are added does *not* impact the sum.

Theorem 16.3. *If (a_n) is absolutely summable, and (b_n) is any rearrangement of (a_n), then*

- *(b_n) is absolutely summable,*
- *$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$, and*
- *$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |b_n|$.*

Proof: Denote by s_n the partial sums of $(a_k)_{k=1}^{\infty}$ (i.e., $s_n = \sum_{k=1}^{n} a_k$) and by t_n the partial sums of (b_n). Let $\ell = \sum_{n=1}^{\infty} a_n$.

[248] "added" in quotes because q_n is negative.
[249] Something basic, coming from conditional convergence of (a_n).

Fix $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $|\ell - s_N| < \varepsilon/2$ (by absolute convergence of (a_n)), and any finite sum from $\{|a_{N+1}|, |a_{N+2}|, \cdots\}$ is at most $\varepsilon/2$ (applying the Cauchy criterion to $(|a_n|)$).

Now because (b_n) is a rearrangement of (a_n), there is an $M \in \mathbb{N}$ (perhaps much larger than M) such that all of $a_1, \ldots, a_N$ appear among $b_1, \ldots, b_M$.

For $m > M$, we have

$$\begin{aligned} |\ell - t_m| &= |\ell - s_N - (t_m - s_N)| \\ &\leq |\ell - s_N| + |t_m - s_N| \\ &< \varepsilon \end{aligned}$$

(the last inequality because $|t_m - s_N|$ a finite sum from among $\{|a_{N+1}|, |a_{N+2}|, \cdots\}$). So $\sum_{n=1}^{\infty} b_n = \ell = \sum_{n=1}^{\infty} a_n$.

To deal with $\sum_{n=1}^{\infty} |b_n|$, note that $(|b_n|)$ is a rearrangement of $(|a_n|)$, and since $(|a_n|)$ absolutely summable, by what we just proved, $\sum_{n=1}^{\infty} |b_n|$ converges to $\sum_{n=1}^{\infty} |a_n|$. $\square$