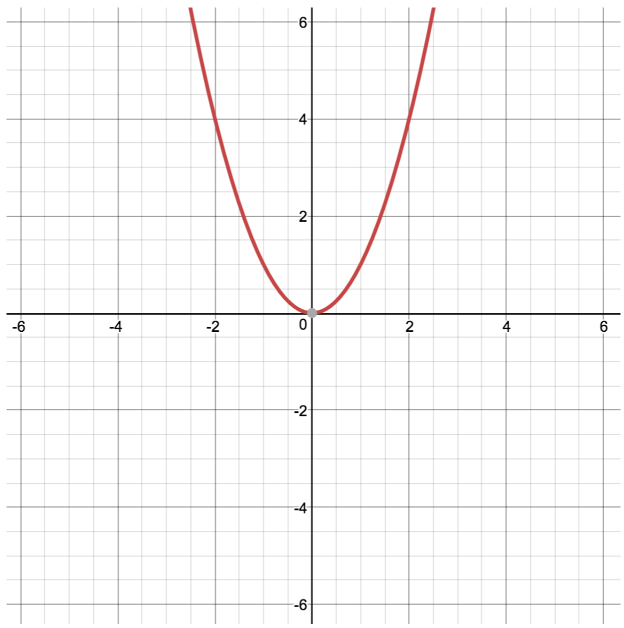
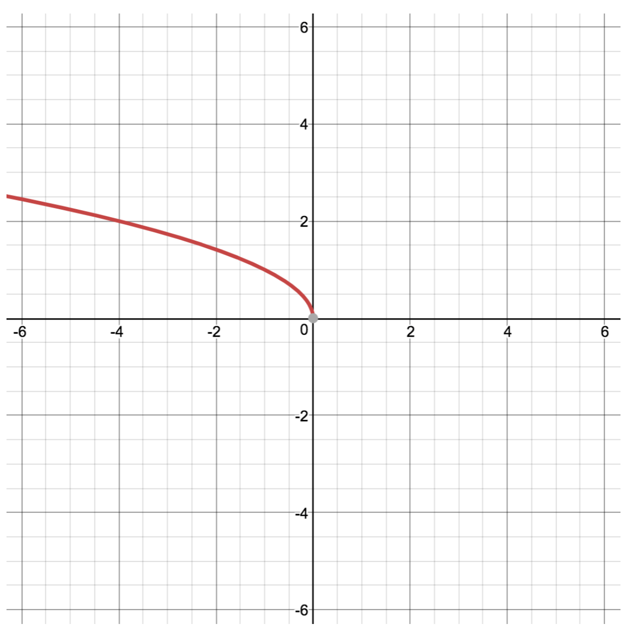
6 Limits

What does a function "look like" as the inputs "approach" a particular input? We'll formalize this vague question, already brought up at the end of the last section, using the notion of a limit. To begin, let us note that there are many possible behaviors a function might exhibit as the inputs approach a particular value a. We illustrate ten possible such behaviors here.

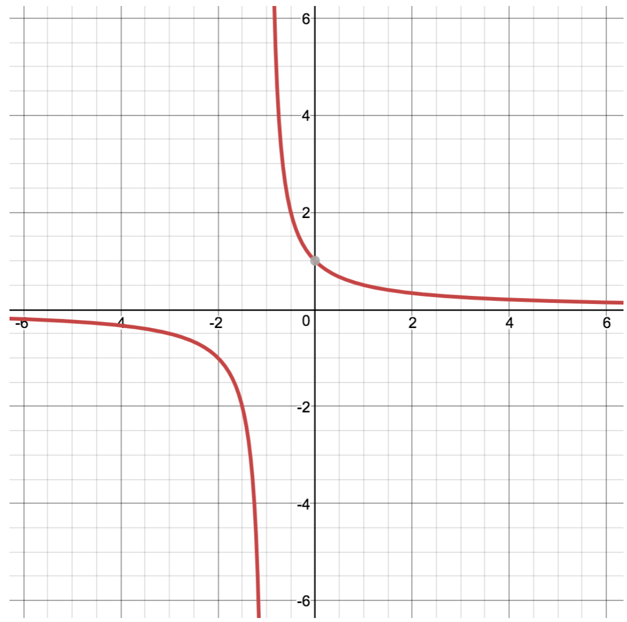
1. Function exhibits no problems at a $f_1(x) = x^2$ at $a = 1$.



2. Function defined nowhere near a $f_2(x) = \sqrt{-x}$ at $a = 1$.

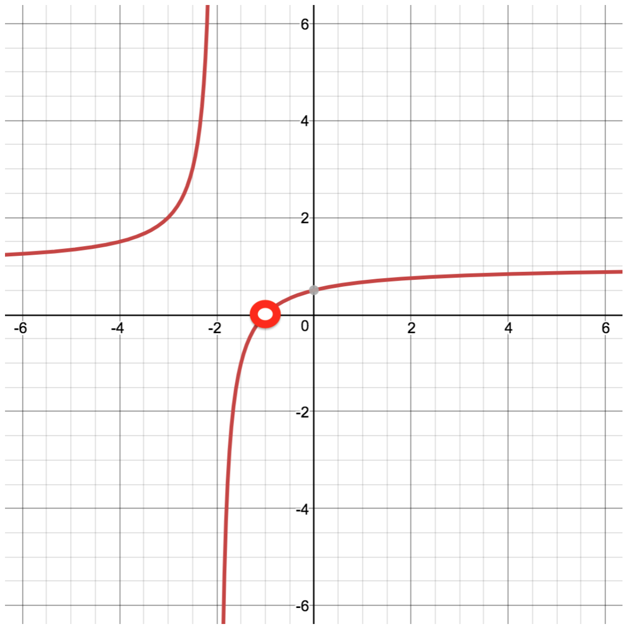


3. Function blows up to infinity approaching a $f_3(x) = 1/(1+x)$ at $a = -1$.

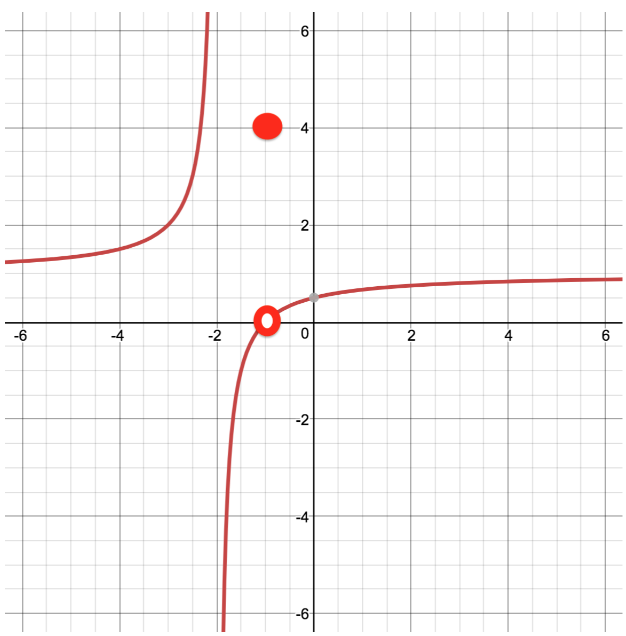


4. Function not defined at a**, but otherwise unremarkable** $f_4(x) = 1/(1 + (1/(1 + x)))$, $a = -1$. This situation, a function with a "hole", might seem odd, but it can arise naturally. Notice here that $f_4 = f_3 \circ f_3$, and that the expression on the right-hand side of the definition of f_4 can be re-written as $(1+x)/(2+x)$, which *does* make sense at $x = -1$. So the function $f_4 = f_3 \circ f_3$, (with natural domain $\mathbb{R} - \{-1, -2\}$), is identical to the function that sends x to $(1+x)/(2+x)$ (which has natural domain $\mathbb{R} - \{-2\}$), *except* at -1, where f_4 has a "naturally occurring" hole.

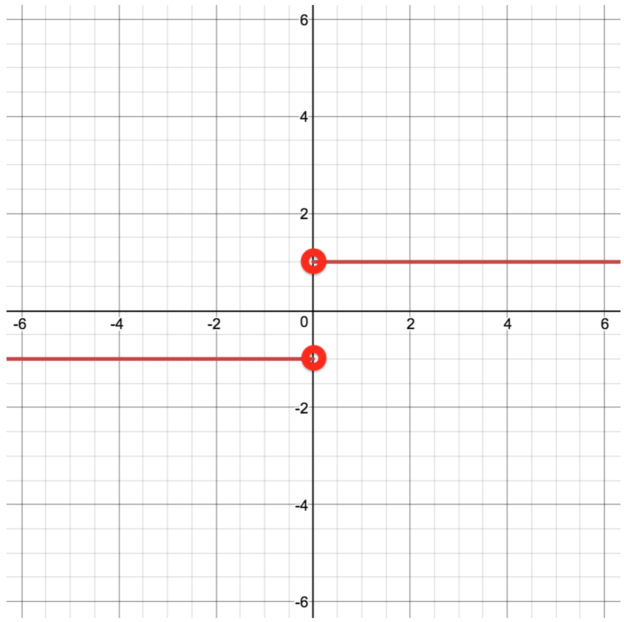
Notice also the graphical notation that we use to indicate the "hole" at -1: literally, a hole.



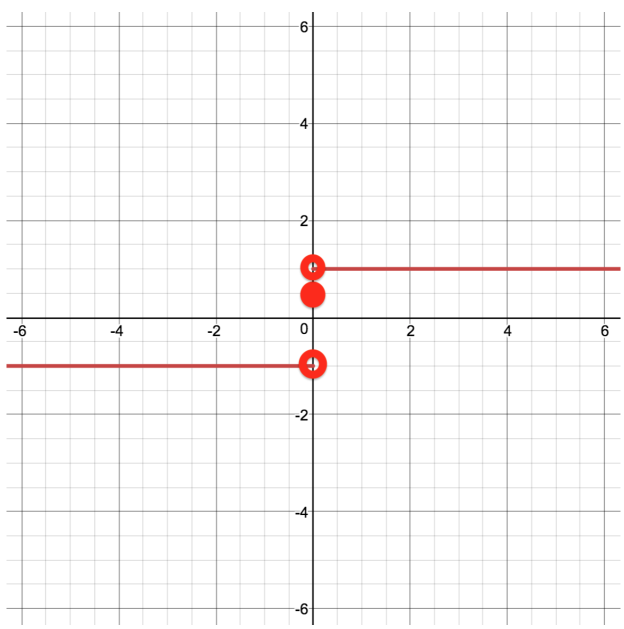
5. Function with "wrong value" at a $f_5(x) = \begin{cases} f_4(x) & \text{if } x \neq -1 \\ 4 & \text{if } x = -1 \end{cases}$ at $a = -1$.



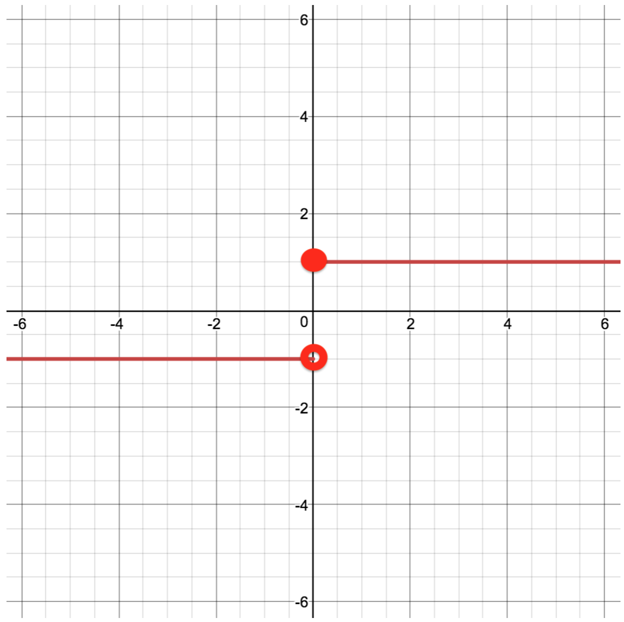
6. Function with a "jump" at a (1) $f_6(x) = \frac{x}{|x|}$ at $a = 0$. The natural domain here is $\mathbb{R} - \{0\}$, and for positive x, $x/|x| = 1$ while for negative x, $x/|x| = -1$. Notice that we graphically indicate the failure of the function to be defined at 0 by *two* holes, one at the end of each of the intervals of the graph that end at 0.



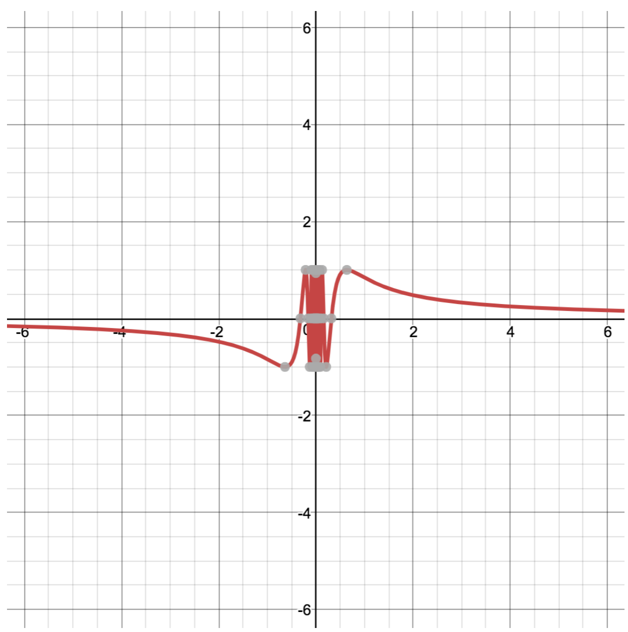
7. Function with a "jump" at a (2) $f_7(x) = \begin{cases} f_6(x) & \text{if } x \neq 0 \\ 1/2 & \text{if } x = 0 \end{cases}$ at $a = 0$. Notice that we graphically indicate the value of the function at 0 with a *solid* holes at the appropriate height.



8. Function with a "jump" at a (3) $f_8(x) = \begin{cases} f_6(x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ at $a = 0$. Notice that here we graphically indicate the behavior of the function around its jump with an appropriate combination of holes and solid holes.



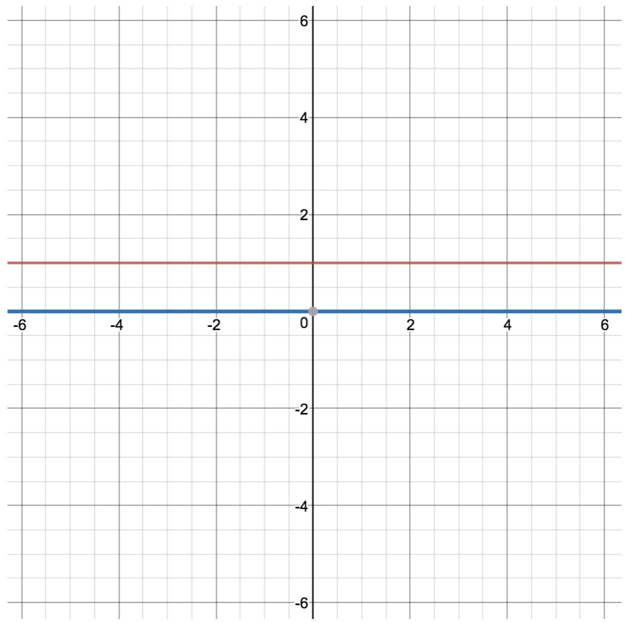
9. Oscillatory function near a $f_9(x) = \sin(1/x)$ at $a = 0$. The natural domain here is $\mathbb{R} - \{0\}$. Notice the complete failure of the graph to convey the behavior of the function!



10. Chaotic function near a $f_{10}(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ at $a = 0$. This function is often called the *Dirichlet* function[73]. Because the rationals are dense in the reals, and so are the irrationals (given any real, there are rationals arbitrarily close to it, and

[73] After the German mathematician Peter Dirichlet, https://en.wikipedia.org/wiki/Peter_Gustav_Lejeune_Dirichlet.

irrationals arbitrarily close to it), the graph of f_{10} just looks like two horizontal lines, and appears to completely fail the vertical line test!



Other behaviors are possible, too, and of course, we could have one kind of behavior on one side of a, and another on the other side.

6.1 Definition of a limit

We would like to develop a definition of the notion "f approaches a limit near a", or "the outputs of f approach a limit, as the inputs approach a", that accounts for our intuitive understanding of the behavior of each of f_1 through f_{10}. Here is an intuitive sense of what is going on in each of the examples:

- f_1 approaches 1 near 1 (as input values get closer to 1, outputs values seem to get closer to 1).
- f_2 doesn't approach a limit near 1 (it isn't even defined near 1).
- f_3 doesn't approach a limit near -1 (or, it approaches some infinite limit — as input values get closer to -1, output values either get bigger and bigger positively, or bigger and bigger negatively).
- Even though f_4 is not defined at -1, it appears that f_4 approaches a limit of 0 near -1 (as input values get closer to -1, outputs values seem to get closer to 0).
- Even though $f_5(-1)$ is not 0, it seems reasonable still to say that f_5 approaches a limit of 0 near -1 (as input values get closer to -1, outputs values seem to get closer to 0).

- f_6 doesn't approach a limit near 0 (as input values get closer to 0 from the right, the outputs values seem to get closer to 1, but as input values get closer to 0 from the left, the outputs values seem to get closer to -1; this ambiguity suggests that we should not declare there to be a limit).

- f_7 doesn't approach a limit near 0 (exactly as f_6: specifying a value for the function at 0 doesn't change the behavior of the function as we approach 0).

- f_8 doesn't approach a limit near 0 (exactly as f_7).

- f_9 doesn't approach a limit near 0 (the outputs oscillate infinitely in the interval $[-1, 1]$ as the inputs approach 0, leading to an even worse ambiguity than that of f_6).

- f_{10} doesn't approach a limit near 0 (the outputs oscillate infinitely between -1 and 1 as the inputs approach 0, again leading to an worse ambiguity than that of f_6).

What sort of definition will capture these intuitive ideas of the behavior of a function, near a potential input value? As a provisional definition, we might take what is often considered the "definition" of a limit:

> **Provisional definition of function tending to a limit**: A function f tends to a limit near a, if there is some number L such that f can be made arbitrarily close to L by taking input values sufficiently close to a.

This definition seems to work fine for f_1 through f_4. For f_4, for example, it seems very clear that we can get the function to take values arbitrarily close to 0, by only considering input values that are pinned to be sufficiently close to -1 (on either side); and for f_3, no candidate L that we might propose for the limit will work — as soon as we start considering inputs that are too close to -1, the values of the outputs will start to be very far from L (they will either have the wrong sign, or have far greater magnitude than L).

It breaks down a little for f_5: we can't make output values of f_5 be arbitrary close to 0 by choosing input values sufficiently close to -1, because -1 surely fits the "sufficiently close to -1" bill (nothing could be closer!), and $f_5(-1) = 4$, far from 0. The issue here is that we want to capture the sense of how the function is behaving *as inputs get close to a*, and so we really should *ignore* what happens *exactly at a*. There's an easy fix for this: add "(not including a itself)" at the end of the provisional definition.

f_6 presents a more serious problem. We can certainly make the outputs of f_6 be arbitrarily close to 1, by taking inputs values sufficiently close to 0 — indeed, *any* positive input value has output *exactly* 1. But by the same token, we can make the outputs of f_6 be arbitrarily close to -1, by taking inputs values sufficiently close to 0 — *any* negative input value has output *exactly* -1.

The issue is that we are "cherry picking" the inputs that are sufficiently close to 0 — positive inputs to get the limit to be 1, negative inputs to get the limit to be -1. In f_9 the situation is even more dramatic. If we pick any L between -1 and 1, we can find a sequence of numbers (one in each oscillation of the function) that get arbitrarily close to 0, such that when f_9 is evaluated at each of these numbers, the values are always *exactly* L (not just getting closer to L) — just look at the infinitely many places where that line $y = L$ cuts across the graph of f_9. So we can make, with our provisional definition, a case for *any* number between -1 and 1 being a limit of the function near 0! This runs at odds to intuition.

We need to remove the possibility of "cherry-picking" values of the input close to a to artificially concoct a limit that shouldn't really be a limit. The way we will do that is best described in terms of a game, played by Alice and Bob.

Suppose Alice and Bob are looking at the function $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto 3x$. Alice believes that as x approaches 1, f approaches the limit 3. Bob is skeptical, and needs convincing. So:

- Bob says "1", and challenges Alice to show that for *all* values of the input sufficiently close to 3, f is within 1 of 9 (asking for *all* values is what eliminates the possibility of cherry-picking values). Think of "1" as a "window of tolerance".

- Alice notices that as x goes between 22/3 and 31/3, $f(x)$ goes between 8 and 10; that is, as long as x is within 1/3 of 3, $f(x)$ is within 1 of 9. So she convinces Bob that output values can be made to be within 1 of 9 by telling him to examine values of x within 1/3 of 1.

- Bob is ok with this, but now wants to see that f can be forced to be even closer to 1. He says "1/10", a smaller window of tolerance, and challenges Alice to show that for all values of the input sufficiently close to 3, f is within 1/10 of 9. Alice repeats her previous calculations with the new challenge number, and responds by saying "1/30": all values of x within 1/30 of 3 give values of $f(x)$ within 1/10 of 9.

- Bob ups the ante, and says "1/1000". Alice responds by saying "1/3000": all values of x within 1/3000 of 3 give values of $f(x)$ within 1/1000 of 9.

- Bob keeps throwing values at Alice, and Alice keeps responding. But Bob won't be fully convinced, until he knows that Alice can make a valid response for *every* possible window of tolerance. So, Bob says "ε: an arbitrary number greater than 0". Now Alice's response must be one that depends on ε, and is such that for each particular choice of $\varepsilon > 0$, evaluates to a valid response. She notices that as x goes between $3 - \varepsilon/3$ and $3 + \varepsilon/3$, $f(x)$ goes between $9 - \varepsilon$ and $9 + \varepsilon$; that is, as long as x is within $\varepsilon/3$ of 3, $f(x)$ is within ε of 9. She tells this to Bob, who is now convinced that as x approaches 1, f approaches the limit 3.

This leads to the definition of a limit.

Definition of function tending to a limit: A function f tends to a limit near a, if

- f is defined near a, meaning that for some small enough number b, the set $(a-b, a+b) \setminus \{a\}$ is in domain of f,

and

- there is some number L such that
- for all positive numbers ε
- there is a positive number δ such that
- whenever x is within a distance δ of a (but is not equal to a)
- f is within ε of L.

More succinctly, f tends to a limit near a, if f is defined near a and there is some number L such that for all $\varepsilon > 0$ there is $\delta > 0$ such that for all x, $0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$.

We write $f(x) \to L$ as $x \to a$ or $\lim_{x\to a} f(x) = L$.

6.2 Examples of calculating limits from the definition

Here's a simple example. Consider the constant function $f(x) = c$ for some real c. It seems clear that for any real a, $\lim_{x\to a} f(x) = c$. To formally verify this, let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - c| < \varepsilon$. But $|f(x) - c| = 0 < \varepsilon$ for *every* x; so we can choose *any* $\delta > 0$ and the implication will be true. In particular, it will be true when we take, for example, $\delta = 1$.

Here's another simple example. Consider the linear function $f(x) = x$. It seems clear that for any real a, $\lim_{x\to a} f(x) = a$. To formally verify this, let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - a| < \varepsilon$. But $|f(x) - a| = |x-a|$; so we are looking for a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|x-a| < \varepsilon$. It is clear that we will succeed in this endeavor by taking $\delta = \varepsilon$; note that since $\varepsilon > 0$, this choice of δ is positive.

The next simplest example is the function $f(x) = x^2$. It seems clear that for any real a, $\lim_{x\to a} f(x) = a^2$. The verification of this from the definition will be considerably more involved than the first two examples.

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|x^2 - a^2| < \varepsilon$. Since the only leverage we have is the choice of δ, and δ is related to $|x-a|$, it seems like it will be very helpful to somehow rewrite $|x^2 - a^2| < \varepsilon$ in a way that brings the expression $|x-a|$ into play. We have such a way, since

$$|x^2 - a^2| = |(x-a)(x+a)| = |x-a||x+a|.$$

We want to make the product of these two things small (less than ε). We can easily make $|x-a|$ small — in fact, we get a completely free hand in choosing how small this term is. We don't get to make $|x+a|$ small, however, and in fact we shouldn't expect to be able to make it small: near a, $|x+a|$ is near $|2a|$, which isn't going to be arbitrarily small.

This is an easily resolved problem. We only need to make $|x+a|$ *slightly* small. We can then use the freedom we have to make $|x-a|$ as small as we want, to make it so small that, even when multiplied by $|x+a|$, the product is still smaller than ε.

Here's a first attempt: as we've said, near a, $|x+a|$ is near $|2a|$, so $|x-a||x+a|$ is near $|2a||x-a|$. So we should make $|x-a|$ be smaller than $\varepsilon/|2a|$, to get $|x-a||x+a|$ smaller than ε.

One problem here is that $|a|$ might be 0, and so we are doing an illegal arithmetic operation. Another problem is that we are vaguely saying that "near a", $|x+a|$ is "close to" $|2a|$, which is not really an acceptable level of precision.

Here's a more rigorous approach: let's start by promising that whatever δ we choose, it won't be bigger than 1 (this is a completely arbitrary choice). With this promise, we know that when $0 < |x-a| < \delta$ we definitely have $|x-a| < 1$, so x is in the interval $(a-1, a+1)$. That means that $x+a$ is in the interval $(2a-1, 2a+1)$. At most how big can $|x+a|$ be in this case? At most the maximum of $|2a-1|$ and $|2a+1|$. By the triangle inequality, $|2a-1| \leq |2a|+1$ and $|2a+1| \leq |2a|+1$, and so, as long as we stick to our promise that $\delta \leq 1$, we have $|x+a| < |2a|+1$. This makes $|x^2-a^2| < (2|a|+1)|x-a|$. We'd like this to be at most ε, so we would like to choose δ to be no bigger than $\varepsilon/(2|a|+1)$ (thus forcing $|x-a| < \varepsilon/(2|a|+1)$ and $|x^2-a^2| < \varepsilon$ whenever $0 < |x-a| < \delta$).

We don't want to simply say "ok, take δ to be any positive number $\leq \varepsilon/(2|a|+1)$" (note that $\varepsilon/(2|a|+1) > 0$, so there *is* such a positive δ). Our choice here was predicated on our promise that $\delta \leq 1$. So what we really want to do, is choose δ to be any positive number no bigger than *both* $\varepsilon/(2|a|+1)$ *and* 1. We can do this, for example, by taking δ to be the minimum of $\varepsilon/(2|a|+1)$ and 1, or, symbolically,

$$\delta = \min\left\{\frac{\varepsilon}{2|a|+1}, 1\right\}.$$

Going back through the argument with this choice of δ, we see that all the boxes are checked: suppose $0 < |x-a| < \delta$. Then in particular we have $|x-a| < 1$, and we also have $|x-a| < \varepsilon/(2|a|+1)$. From $|x-a| < 1$ we deduce $a-1 < x < a+1$, so $2a-1 < x+a < 2a+1$, so $|x+a| < \max\{|2a-1|, |2a+1|\} \leq 2|a|+1$. From this and $|x-a| < \varepsilon/(2|a|+1)$ we deduce

$$|x^2-a^2| = |x+a||x-a| < \frac{\varepsilon}{2|a|+1}(2|a|+1) = \varepsilon,$$

and so, since ε was arbitrarily, we deduce that indeed $\lim_{x\to a} f(x) = a^2$.

We do one more example: $\lim_{x\to 2} \frac{3}{x}$. It seems clear that this limit should be $3/2$. Given $\varepsilon > 0$, we need $\delta > 0$ such that $0 < |x-2| < \delta$ implies $|(3/x) - (3/2)| < \varepsilon$. We have

$$\left|\frac{3}{x} - \frac{3}{2}\right| = \left|\frac{6-3x}{2x}\right| = \frac{3}{2}\frac{|x-2|}{|x|}.$$

We want to make this small, which requires making $|x|$ *large*. If $\delta \leq 1$ then $0 < |x-2| < \delta$ implies $x \in (1,3)$, so $|x| > 1$ and $3/(2|x|) < 3/2$. So if both $\delta \leq 1$ *and* $\delta \leq 2\varepsilon/3$, we have

$$\left|\frac{3}{x} - \frac{3}{2}\right| = \frac{3}{2}\frac{|x-2|}{|x|} < \frac{3}{2} \cdot \frac{2\varepsilon}{3} = \varepsilon$$

as long as $0 < |x-2| < \delta$. Taking δ to be $\min\{1, 2\varepsilon/3\}$ verifies $\lim_{x\to 2} 3/x = 3/2$.

Notice that we initially choose $\delta \leq 1$ to get a lower bound on $|x|$. Any δ would have worked, as long as we avoided have 0 in the possible range of values for x (if we allowed 0 to be in the possible range of values for x we would have *no* upper bound on $1/|x|$).

Essentially all examples of proving claimed values of limits directly from the definition follow the path of these last two examples:

- do some algebraic manipulation on the expression $|f(x)-L|$ to isolate $|x-a|$ (a quantity we have complete control over);
- by putting a preliminary bound on δ, put some bound $B > 0$ on the part of $|f(x)-L|$ that does not involve $|x-a|$;
- choose δ to be the smaller of ε/B and the preliminary bound on δ.

6.3 Limit theorems

To streamline the process of computing limits, we prove a few general results. The first is a result that says that the limits of sums, products and ratios of functions, are the sums, products and ratios of the corresponding limits.

Theorem 6.1. *(Sum/product/reciprocal theorem) Let f, g be functions both defined near some a. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ (that is, both limits exist, and they take the claimed values). Then*

- $\lim_{x\to a}(f+g)(x)$ *exists and equals* $L+M$*;*
- $\lim_{x\to a}(fg)(x)$ *exists and equals* LM*; and,*
- *if* $M \neq 0$ *then* $\lim_{x\to a}(1/g)(x)$ *exists and equals* $1/M$*.*

Proof: We begin with the sum statement. Since f, g are defined near a, so is $f+g$. Let $\varepsilon > 0$ be given. Because $\lim_{x\to a} f(x) = L$, there is $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies $|f(x)-L| < \varepsilon/2$, and because $\lim_{x\to a} g(x) = M$, there is $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ implies $|g(x)-M| < \varepsilon/2$. Now if $\delta = \min\{\delta_1, \delta_2\}$, we have that if $0 < |x-a| < \delta$ then

$$\begin{aligned} |(f+g)(x)-(L+M)| &= |(f(x)+g(x))-(L+M)| \\ &= |(f(x)-L)+(g(x)-M)| \\ &\leq |f(x)-L|+|g(x)-M| \quad \text{by triangle inequality} \\ &< \varepsilon/2+\varepsilon/2 = \varepsilon. \end{aligned}$$

This shows that $\lim_{x\to a}(f+g)(x) = L+M$.

We now move on to the product statement, which is a little more involved. Again, since f, g are defined near a, so is fg. Let $\varepsilon > 0$ be given. We have[74]

$$\begin{aligned} |(fg)(x) - LM| &= |f(x)g(x) - LM)| \\ &= |f(x)g(x) - Lg(x) + Lg(x) - LM)| \\ &= |g(x)(f(x) - L) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - M| \quad \text{by triangle inequality.} \end{aligned}$$

We can make $|f(x)-L|$ and $|g(x)-M|$ as small as we like; we would like to make them small enough that $|g(x)||f(x) - L| < \varepsilon/2$ and $|L||g(x) - M| < \varepsilon/2$. The second of those is easy to achieve. There's $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies $|g(x) - M| < \varepsilon/(2(|L| + 1))$, so $|L||g(x) - M| < |L|(\varepsilon/(2(|L| + 1))) < \varepsilon/2$.[75]

The first is less easy. We need an upper bound on $|g(x)|$. We know that there is a $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies $|g(x) - M| < 1$ so $|g(x)| < |M| + 1$. There's also a $\delta_3 > 0$ such that $0 < |x - a| < \delta_3$ implies $|f(x) - L| < \varepsilon/(2(|M| + 1))$.

As long as δ is at most the minimum of δ_1, δ_2 and δ_3, we have that $0 < |x - a| < \delta$ implies all of

- $|L||g(x) - M| < \varepsilon/2$
- $|g(x)| < |M| + 1$, so $|g(x)||f(x) - L| < (|M| + 1)||f(x) - L||$
- $|f(x) - L| < \varepsilon/(2(|M| + 1))$, so $|g(x)||f(x) - L| < \varepsilon/2$,
- so, combining first and fourth points, $|g(x)||f(x) - L| + |L||g(x) - M| < \varepsilon$.

It follows from the chain of inequalities presented at the start of the proof that $0 < |x-a| < \delta$ implies

$$|(fg)(x) - LM| < \varepsilon,$$

and so $\lim_{x\to a}(fg)(x) = LM$.

We now move on to the reciprocal statement. Here we have to do some initial work, simply to show that $(1/g)$ is defined near a. To show this, we need to establish that near a, g is not 0. The fact that g approaches M near a, and $M \neq 0$, strongly suggests that this is the case. To verify it formally, we make (and prove) the following general claim, that will be of some use to us in the future.

[74] We use a trick here — adding and subtracting the same quantity. The motivation is that we want to introduce $|f(x) - L|$ into the picture, so we subtract $Lg(x)$ from $f(x)g(x)$. But to maintain equality, we then need to add $Lg(x)$; this conveniently allows us to bring $|g(x) - M|$ into the picture, also. We'll see this kind of trick many times.

[75] Why did we want $2(|L| + 1)$ in the denominator, rather than $2|L|$? This was an overkill designed to avoid the possibility of dividing by 0.

Claim 6.2. *Let g be defined near a, and suppose $\lim_{x\to a} g(x)$ exists and equals M. If $M > 0$, then there is some δ such that $0 < |x-a| < \delta$ implies $g(x) \geq M/2$. If $M < 0$, then there is some δ such that $0 < |x-a| < \delta$ implies $g(x) \leq M/2$. In particular, if $M \neq 0$ then there is some δ such that $0 < |x-a| < \delta$ implies $|g(x)| \geq |M|/2$ and $g(x) \neq 0$.*

Proof of claim: Suppose $M > 0$. Applying the definition of $\lim_{x\to a} g(x) = M$ with $\varepsilon = M/2$ we find that there is some δ such that $0 < |x-a| < \delta$ implies $|g(x) - M| < M/2$, which in turn implies $g(x) \geq M/2$. On the other hand, if $M < 0$, then applying the definition of $\lim_{x\to a} g(x) = M$ with $\varepsilon = -M/2$ we find that there is some δ such that $0 < |x-a| < \delta$ implies $|g(x) - M| < -M/2$, which in turn implies $g(x) \leq -M/2$. □

We have established that $1/g$ is defined near a, and in fact that if $M > 0$ then g is positive near a, while if $M < 0$ then g is negative near a. We next argue that $\lim_{x\to a}(1/g)(x) = 1/M$. Given $\varepsilon > 0$, choose $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies $|g(x)| \geq |M|/2$ (which we can do by the claim). We have

$$\begin{aligned}
\left|\left(\frac{1}{g}\right)(x) - \frac{1}{M}\right| &= \left|\frac{1}{g(x)} - \frac{1}{M}\right| \\
&= \left|\frac{M - g(x)}{Mg(x)}\right| \\
&= \frac{|g(x) - M|}{|M||g(x)|} \\
&\leq \frac{2}{|M|^2}|g(x) - M|.
\end{aligned}$$

We would *like* to make $|(1/g)(x) - (1/M)| < \varepsilon$. One way to do this is to force $(2/|M|)^2|g(x) - M|$ to be smaller than ε, that is, to force $|g(x) - M|$ to be smaller than $(|M|^2\varepsilon)/2$.

Since $g \to M$ as $x \to a$, and since $(|M|^2\varepsilon)/2 > 0$, there is a $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ indeed implies $|g(x) - M| < (|M|^2\varepsilon)/2$.

So, if we let δ be the smaller of δ_1 and δ_2 then $0 < |x-a| < \delta$ implies $|(1/g)(x) - 1/M| < \varepsilon$, so that indeed $\lim_{x\to a}(1/g)(x) = 1/M$. □

An obvious corollary of the above is the following, which we give a proof of as a prototype of proofs of this kind.

Corollary 6.3. *For each $n \geq 1$, let $f_1, \ldots, f_n$ be functions all defined near some a. Suppose that $\lim_{x\to a} f_i(x) = L_i$ for each $i \in \{1, \ldots, n\}$. Then*

- *$\lim_{x\to a}(f_1 + \cdots + f_n)(x)$ exists and equals $L_1 + \cdots + L_n$.*

Proof: We proceed by induction on n, with the base case $n = 1$ trivial (it asserts that if $\lim_{x\to a} f_1(x) = L_1$ then $\lim_{x\to a} f_1(x) = L_1$).

For the induction step, suppose the result is true for some $n \geq 1$, and that we are given $n+1$ functions $f_1, \ldots, f_{n+1}$, all defined near a, with $f_i \to L_i$ near a for each i. We have

$$\lim_{x\to a}(f_1 + \cdots + f_n)(x) = L_1 + \cdots + L_n$$

by the induction hypothesis, and $\lim_{x\to a} f_{n+1}(x) = L_{n+1}$ by hypothesis of the corollary. By the sum/product/reciprocal theorem, we have that $\lim_{x\to a}((f_1+\cdots+f_n)+f_{n+1})(x)$ exists and equals $(L_1+\cdots+L_n)+L_{n+1}$; but since $((f_1+\cdots+f_n)+f_{n+1})(x) = (f_1+\cdots+f_{n+1})(x)$ and $(L_1+\cdots+L_n)+L_{n+1} = L_1+\cdots+L_n+L_{n+1}$, this immediately says that

$$\lim_{x\to a}(f_1+\cdots+f_{n+1})(x) = L_1+\cdots+L_{n+1}.$$

The corollary is proven, by induction.[76] □

We may similarly prove that for each $n \geq 1$, if $f_1,\ldots,f_n$ are functions all defined near some a, and if $\lim_{x\to a} f_i(x) = L_i$ for each $i \in \{1,\ldots,n\}$, then

- $\lim_{x\to a}(f_1\cdot\cdots\cdot f_n)(x)$ exists and equals $L_1\cdot\cdots\cdot L_n$.

This has an important consequence. Starting from the basic results that for any a, c, $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$, by repeated applications of the sum/product/reciprocal theorem, together with its corollaries, we obtain the following important labor-saving results:

- Suppose that P is a polynomial. Then for any a, $\lim_{x\to a} P(x)$ exists and equals $P(a)$.
- Suppose that R is a rational function, say $R = P/Q$ where P, Q are polynomials. If a is in the domain of R, that is, if $Q(a) \neq 0$, then $\lim_{x\to a} R(x)$ exists and equals $R(a)$, that is,
$$\lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

For example, we can immediately say

$$\lim_{x\to 1}\frac{2x^2-4x}{x^3-8} = \frac{2(1)^2-4(1)}{(1)^3-8} = -\frac{2}{7},$$

piggy-backing off our general theorems, and avoiding a nasty derivation from first principles.

What about $\lim_{x\to 2}(2x^2-4x)/(x^3-8)$? Here a direct evaluation is not possible, because 2 is not in the domain of $(2x^2-4x)/(x^3-8)$. But because 2 is not in the domain, we can algebraic manipulate $(2x^2-4x)/(x^3-8)$ by dividing above and below the line by $x-2$ — this operation is valid exactly when $x \neq 2$! Formally we can say

$$\frac{2x^2-4x}{x^3-8} = \frac{2x(x-2)}{(x-2)(x^2+2x+4)} = \frac{2x}{x^2+2x+4},$$

valid on the entire domain of $(2x^2-4x)/(x^3-8)$. So

$$\lim_{x\to 2}\frac{2x^2-4x}{x^3-8} = \lim_{x\to 2}\frac{2x}{x^2+2x+4} = \frac{4}{14} = \frac{2}{7}.$$

[76] Notice that in the induction step, dealing with deducing $p(n+1)$ from $p(n)$, we needed to invoke the $n=2$ case. This occurs frequently when extending a result concern two objects to the obvious analog result concerning many objects. Examples include the general distributive law, and the general triangle inequality.

One last note on the limit. We have been implicitly assuming throughout all of this section that if f approaches a limit L near a, then L is the *only* limit that it approaches. We can easily prove this.

Claim 6.4. *Suppose f is defined near a and that $\lim_{x \to a} f(x) = L$, and also $\lim_{x \to a} f(x) = M$. Then $L = M$.*

Proof: Suppose for a contradiction that $L \neq M$. Assume, without any loss of generality[77], that $L > M$. Set $\varepsilon = (L - M)/4$. There is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$ and $|f(x) - M| < \varepsilon$. The first of these inequalities says that $f(x) > L - \varepsilon$, and the second says $f(x) < M + \varepsilon$, so together they imply that $L - \varepsilon < M + \varepsilon$, or $L - M < 2\varepsilon$, or $(L - M)/4 < \varepsilon/2$, or $\varepsilon < \varepsilon/2$, a contradiction. We conclude that $L = M$. □

6.4 Non-existence of limits

What does it mean for a function f *not* to tend to a limit L near a? For a function f to tend to a limit L near a, two things must happen:

1. f must be defined near a, and
2. for all $\varepsilon > 0$ there is $\delta > 0$ such that for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

So for f not to tend to L, *either* the first clause above fails, so f is not defined near a, or the second clause fails. To understand what it means for the second clause to fail, it's helpful to write it symbolically, and then use the methods we have discussed earlier to negate it. The clause is

$$(\forall \varepsilon)(\exists \delta)(\forall x)((0 < |x - a| < \delta) \Rightarrow (|f(x) - L| < \varepsilon))$$ [78]

and its negation is

$$(\exists \varepsilon)(\forall \delta)(\exists x)((0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \varepsilon)).$$

So, unpacking all this, we get:

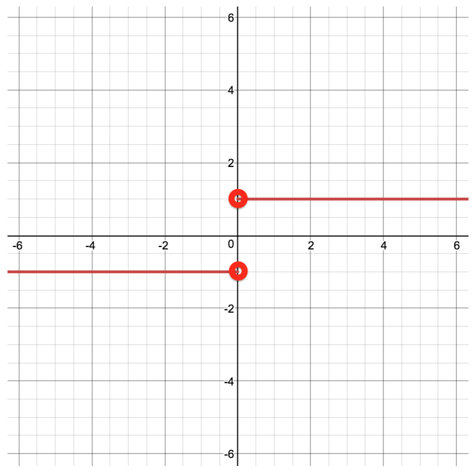
> **Definition of a function not tending to a limit L near a**: f does not approach the limit L near a if either
>
> - f is not defined near a (meaning, in any open interval that includes a, there are points that are not in the domain of f)
>
> or

[77] A handy phrase, but one to be used only when you are really saying that no generality is lost.

[78] Notice that we have included the quantification $\forall x$. Without this, the clause would be a predicate (depending on the variable x), rather than a statement.

- there's an $\varepsilon > 0$ (a window of tolerance around L presented by Bob) such that
- for all $\delta > 0$ (no matter what window of tolerance around a that Alice responds with)
- there is an x with $0 < |x - a| < \delta$ (an $x \neq a$ that is within δ of a)
- but with $|f(x) - L| \geq \varepsilon$ ($f(x)$ is at least ε away from L).

As an example, consider the function f_6 defined previously, that is given by $f_6(x) = x/|x|$.



It's seems quite clear that f_6 does not approach a limit near 0; the function gets close to both 1 and -1 in the vicinity of 0, so there isn't a single number that the function gets close to (and we know that if the limit exists, it is unique).

We use the definition just given of a function *not* tending to a limit, to verify that $\lim_{x\to 0} f_6(x) \neq 3/4$. Take $\varepsilon = 1/10$ (this is fairly arbitrary). Now consider any $\delta > 0$. We need to show that there is an $x \neq 0$, in the interval $(-\delta, \delta)$, with $|f_6(x) - 3/4| \geq 1/10$. There are many such x's that work. For example, consider $x = \delta/2$; for this choice of x, $|f_6(x) - 3/4| = |(\delta/2)/(|\delta/2|) - 3/4| = |1 - 3/4| = |1/4| = 1/4 \geq 1/10$[79]

Why did we choose $\varepsilon = 1/10$? We intuited that output values of f_6 could be made arbitrarily close to 1 by cherry-picking values of x close to 0. So to show that values of the output can't be made *always* arbitrarily close to 3/4 by choosing values of the input close enough to 0, we choose an ε so that the interval $(3/4 - \varepsilon, 3/4 + \varepsilon)$ did not get too close to 1 — that allowed us to choose an x close to 0 for which $f_6(x)$ was not close to 3/4. Any ε less that 1/4 would have worked.[80]

More generally, what does it mean for f not to tend to *any* limit near a? It means that for every L, f does not tend to limit L near a.

[79]We could have equally well picked $x = -\delta/2$; then $|f_6(x) - 3/4| = 7/4 \geq 1/10$.

[80]In fact, any ε less than 11/4 would have worked — we could have noticed that output values of f_6 could be made arbitrarily close to -1 by cherry-picking values of x close to 0.

Definition of a function not tending to a limit near a: f does not approach a limit near a if for *every* L it is the case that f does not approach the limit L near a.

Going back to our previous example: we claim that $\lim_{x\to 0} f_6(x)$ does not exist. Indeed, suppose that L is given, and proposed as a (the) limit. We want to find an $\varepsilon > 0$ such that for any $\delta > 0$, we can find at least one value of $x \neq 0$ that is within δ of 0, but that $f_6(x)$ is not within ε of L. We notice that by cherry-picking values of x arbitrarily close to 0, we can get $f_6(x)$ arbitrarily close to *both* -1 *and* to 1. This suggests the following strategy:

- If $L \geq 0$: take $\varepsilon = 1/2$. Given $\delta > 0$, consider $x = -\delta/2$. That's certainly within δ of 0 (and is certainly not equal to 0). But $f_6(x) = -1$, so $f_6(x)$ is distance at least 1 from L, and so not distance less than 1/2.

- If $L < 0$: again take $\varepsilon = 1/2$. Given $\delta > 0$, consider $x = \delta/2$. It's non-zero and within δ of 0, but $f_6(x) = 1$, so $f_6(x)$ is distance more than 1 from L, and so not distance less than 1/2.

One more example: we claim that $\lim_{x\to 0} |\sin(1/x)|$ does not exist. The intuition behind this is the same as for the previous example: by cherry picking values of x, we can get $\sin(1/x)$ to take the value 1, arbitrarily close to 0, and we can get it to take the value 0. Specifically, $|\sin(1/x)|$ takes the value 1 at $1/x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$, so at $x = \pm 2/\pi, \pm 2/3\pi, \pm 2/5\pi, \ldots$, or more succinctly at $x = \pm 2/((2n+1)\pi)$, $n = 0, 1, 2, 3, \ldots$; and $|\sin(1/x)|$ takes the value 0 at $1/x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$, so at $x = \pm 1/(n\pi)$, $n = 0, 1, 2, 3, \ldots$. So, given L (a proposed limit for $|\sin(1/x)|$ near 0), we can again treat two cases, depending on whether L is far from 0 or far from 1.

- If $L \geq 1/2$: take $\varepsilon = 1/4$. Given $\delta > 0$, there is some n large enough that $x := 1/(n\pi)$ is in the interval $(-\delta, \delta)$[81] (and is non-zero). For this x, $|\sin(1/x)| = 0$, which is *not* in the interval $(L - 1/4, L + 1/4)$.

- If $L < 1/2$: again take $\varepsilon = 1/4$. Given $\delta > 0$, there is some n large enough that $x := 2/((2n+1)\pi)$ is in the interval $(-\delta, \delta)$ (and is non-zero). For this x, $|\sin(1/x)| = 1$, which is *not* in the interval $(L - 1/4, L + 1/4)$.

We conclude that $\lim_{x\to 0} |\sin(1/x)|$ does not exist.

In the homework, you'll deal with another situation where a limit doesn't exist: where the output values don't approach a specific value, because they get arbitrarily large in magnitude near the input. We'll return to these "infinite limits" later.

One last comment for the moment about limits not existing: while $\lim_{x\to 0} |\sin(1/x)|$ does not exist, the superficially similar $\lim_{x\to 0} x|\sin(1/x)|$ does, and it's easy to prove that it takes the value 0. Indeed, given $\varepsilon > 0$, take $\delta = \varepsilon$. If $0 < |x| < \delta$ then $|x|\sin(1/x)|| \leq |x| < \delta = \varepsilon$,

[81] Is there???

so the limit is 0. This illustrates that while oftentimes computing limits directly from the definition is a slog, it can sometimes be surprisingly easy.

There's a general phenomenon that this last example — $f(x) = x|\sin(1/x)|$ near 0 — is a special case of. The function $f(x) = x|\sin(1/x)|$ is "squeezed" between two other functions that are quite easy to understand. If g_ℓ, g_u are defined by

$$g_\ell(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

and

$$g_u(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

then we easily have that

$$g_\ell(x) \leq f(x) \leq g_u(x)$$

for all real x. Indeed, for $x \geq 0$ we have, using $0 \leq |\sin(1/x)| \geq 1$, that $0 \leq x|\sin(1/x)| \leq x$, while if $x < 0$ then $0 \leq |\sin(1/x)| \geq 1$ implies $0 \geq x|\sin(1/x)| \geq x$ or $x \leq x|\sin(1/x)| \leq 0$; and these two inequalities together say that $g_\ell(x) \leq f(x) \leq g_u(x)$.

We also have that $g_\ell \to 0$ near 0, and that $g_u \to 0$ near 0. We verify the first of these now (the second is left as an exercise). Given $\varepsilon > 0$ we seek $\delta > 0$ so that $x \in (-\delta, \delta)$ (and $x \neq 0$) implies $g_\ell(x) \in (-\varepsilon, \varepsilon)$. Consider $\delta = \varepsilon$. If non-zero x is in $(-\delta, \delta)$ and is negative, then $g_\ell(x) = x \in (-\delta, \delta) = (-\varepsilon, \varepsilon)$, while if it is positive then $g_\ell(x) = 0 \in (-\delta, \delta) = (-\varepsilon, \varepsilon)$. This shows that $g_\ell \to 0$ near 0.

If both g_ℓ and g_u are approaching 0 near 0, and f is sandwiched between g_ell and g_u, then it should come as no surprise that f is *forced* to approach 0 (the common limit of its upper and lower bounds) near 0. The general phenomenon that this example illustrates is referred to as a *squeeze theorem*.

Theorem 6.5. *(Squeeze theorem) Let f, g, h be three functions, and let a be some real number. Suppose that $f, g,$ are all defined near a, that is, that there is some number $\Delta > 0$ such that on the interval $(a - \Delta, a + \Delta)$ it holds that $f(x) \leq g(x) \leq h(x)$ (except possibly at a, which might or might not be in the domains of any of the three functions). Suppose further that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ both exist and both equal L. Then $\lim_{x\to a} g(x)$ exists and equals L.*

You will be asked for a proof of this in the homework.

6.5 One-sided limits

When discussing the squeeze theorem we saw the function

$$g_u(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0, \end{cases}$$

defined by cases, with different behavior to the right and left of 0 on the number line. When establishing $\lim_{x\to 0} g_u(x)$ we need to consider separately what happens for positive x and negative x. This strongly suggests that there could be some value in a refinement of the definition of limit, that considers separately what happens for x values that are larger a, and smaller than a. The natural refinement is referred to as a *one-sided limit*.

Definition of f approaching L near a *from the right* or *from above*: A function f approaches a limit L from the right near a from the right (or from above)[82] if

- f is defined near a, to the right, meaning that there is some $\delta > 0$ such that all of $(a, a + \Delta)$ is in the domain of f,

and

- for all $\varepsilon > 0$ there is $\delta > 0$ such that $0 < x - a < \delta$ implies $|f(x) - L| < \varepsilon$; that is, whenever x is within δ of a, and x is greater than a ("above" a in magnitude, "to the right of" a on the number line), then $f(x)$ is within ε of L.

We write

- $\lim_{x\to a^+} f(x) = L$, or $\lim_{x\searrow a} f(x) = L$
- $f \to L$ (or $f(x) \to L$) as $x \to a^+$ (or as $x \searrow a$).

Definition of f approaching L near a *from the left* or *from below*: A function f approaches a limit L near a from the left (or from below) if

- f is defined near a, to the left, meaning there is $\delta > 0$ with $(a - \Delta, a)$ in the domain of f,

and

- for all $\varepsilon > 0$ there is $\delta > 0$ such that $-\delta < x - a < 0$ implies $|f(x) - L| < \varepsilon$; that is, whenever x is within δ of a, and x is less than a ("below" a in magnitude, "to the left of" a on the number line), then $f(x)$ is within ε of L.

We write

- $\lim_{x\to a^-} f(x) = L$, or $\lim_{x\nearrow a} f(x) = L$
- $f \to L$ (or $f(x) \to L$) as $x \to a^-$ (or as $x \nearrow a$).

As an example consider the familiar old function $f_6(x) = x/|x|$. We know that $\lim_{x\to 0} f_6(x)$ does not exist. But this coarse statement seems to miss something about f_6 — that the function seems to approach limit 1 near 0, if we are only looking at positive inputs, and seems to approach limit -1 near 0, if we are only looking at negative inputs.

[82] Note well: as you'll see from the definition, it is a that is being approached from above, not L

The notion of one-sided limits just introduced captures this. We claim that $\lim_{x\to 0^+} f_6(x)$ exists, and equals 1. Indeed, given $\varepsilon > 0$, take $\delta = 1$. if $0 < x - 0 < \delta$ then $x > 0$ so $f_6(x) = 1$, and so in particular $|f_6(x) - 1| = 0 < \varepsilon$. Similarly, it's easy to show $\lim_{x\to 0^-} f_6(x) = -1$.

This example shows that both the one-sided limits can exist, while the limit may not exist. It's also possible for one one-sided limit to exist, but not the other (consider the function which takes value $\sin(1/x)$ for positive x, and 0 for negative x, near 0), or for both not to exist (consider $\sin(1/x)$ near 0). So, in summary, if the limit doesn't exist, then at least three things can happen with the one-sided limits:

- both exist, but take different values,
- one exists, the other doesn't, or
- neither exists.

There's a fourth possibility, that both one-sided limits exist and take the same value. But that *can't* happen when the limit does not exist, as we are about to see; and as we are also about to see, if the limit exists then there is one one possibility for the two one-sided limits, namely that they both exist and are equal.

Theorem 6.6. *For a f be a function defined near a, $\lim_{x\to a} f(x)$ exists and equals L if and only if both of $\lim_{x\to a+} f(x)$, $\lim_{x\to a-} f(x)$ exist and equal L.*

Proof: Suppose $\lim_{x\to a} f(x)$ exists and equals L. Let $\varepsilon > 0$ be given. There is $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. In particular that means that $0 < x - a < \delta$ implies $|f(x) - L| < \varepsilon$, so that $\lim_{x\to a+} f(x)$ exists and equal L, and $-\delta < x - a < 0$ implies $|f(x) - L| < \varepsilon$, so that $\lim_{x\to a-} f(x)$ exists and equal L.

Conversely, both of $\lim_{x\to a+} f(x)$, $\lim_{x\to a-} f(x)$ exist and equal L. Given $\varepsilon > 0$ there is $\delta_1 > 0$ such that $0 < x - a < \delta_1$ implies $|f(x) - L| < \varepsilon$, and there is $\delta_2 > 0$ such that $-\delta_2 < x - a < 0$ implies $|f(x) - L| < \varepsilon$. If $\delta = \min\{\delta_1, \delta_2\}$ then $0 < |x - a| < \delta$ implies that either $0 < x - a < \delta \leq \delta_1$, or $-\delta_2 \leq -\delta < x - a < 0$. In either case $|f(x) - L| < \varepsilon$, so $\lim_{x\to a} f(x)$ exists and equal L. □

6.6 Infinite limits, and limits at infinity

A minor deficiency of the real numbers, is the lack of an "infinite" number. The need for such a number can be seen from a very simple example. We have that

$$\lim_{x\to 0} \frac{1}{x^2} \text{ does not exist,}$$

but not because the expression $1/x^2$ behaves wildly near 0. On the contrary, it behaves very predictably: the closer x gets to zero, from either the positive or the negative side, the larger (more positive) $1/x^2$ gets, without bound. It would be helpful to have an "infinite" number, one that is larger than all positive numbers; such a number would be an ideal candidate for the limit of $1/x^2$ near 0.

There is no such real number. But it useful to introduce a symbol that can be used to encode the behavior of expressions like $\lim_{x\to 0} 1/x^2$.

Definition of an infinite limit Say that f *approaches the limit infinity*, or *plus infinity*, near a, denoted

$$\lim_{x\to a} f(x) = \infty$$ [83]

(or sometimes $\lim_{x\to a} f(x) = +\infty$) if f is defined near a, and if

- for all real numbers M
- there is $\delta > 0$
- such that for all real x,

$$0 < |x-a| < \delta \quad \text{implies} \quad f(x) > M.$$ [84]

Similarly, say that f *approaches the limit minus infinity* near a, denoted

$$\lim_{x\to a} f(x) = -\infty$$

if f is defined near a, and if for all real numbers M there is $\delta > 0$ such that for all real x,

$$0 < |x-a| < \delta \quad \text{implies} \quad f(x) < M.$$

Before doing an example, we make the following labor-saving observation. Suppose that we are trying to show $\lim_{x\to a} f(x) = \infty$, and that, for some M_0, we have found $\delta_0 > 0$ such that $0 < |x-a| < \delta_0$ implies $f(x) > M_0$. Then for *any* $M \leq M_0$ we have that $0 < |x-a| < \delta_0$ implies $f(x) > M$. The consequence of this is that in attempting to prove $\lim_{x\to a} f(x) = \infty$, we can start by picking an arbitrary real M_0, and then only attempt to verify the condition in the definition for $M \geq M_0$; this is enough to establish the limit statement. Often in practice, this observation is employed by assuming that $M > 0$, which assumption allows us to divide or multiply an inequality by M without either flipping the direction of the inequality, or having to worry about dividing by 0.

A similar observation can be made about showing $\lim_{x\to a} f(x) = -\infty$ (we need only verify the condition for all $M \leq M_0$; in practice this is often $M < 0$), and analogous observations can be made for establishing one-sided infinite limits (see below).

[83]The symbol "∞" here is just that — a *symbol*. It is not, **not**, a *number*. It has *no* place in any arithmetic calculation involving real numbers!

[84]Note that this is saying that $f(x)$ can be forced to be arbitrarily large and positive, by taking values of x sufficiently close to a.

Now we move on to an example, $\lim_{x\to 0} 1/x^2$. We claim that this limit is plus infinity. Indeed, let $M > 0$ be given. We would like to exhibit a $\delta > 0$ such that $0 < |x| < \delta$ implies $1/x^2 > M$. Now because x and M are both positive, we have that

$$1/x^2 > M \text{ is equivalent to } x^2 < 1/M, \text{ which is equivalent to } |x| < 1/\sqrt{M}.$$

So we may simply take $\delta = 1/\sqrt{M}$ (which is positive).

As with the ordinary limit definition, it is sometimes very helpful to be able to consider separately what happens as we approach a from each of the two possible sides.

Definitions of one-sided infinite limits Say that f approaches the limit (plus) infinity near a *from above*, or *from the right*, denoted

$$\lim_{x\to a^+} f(x) = (+)\infty$$

if f is defined near a from above (in some interval $(a, a+\delta)$, $\delta > 0$), and if

- for all real numbers $M > M_0$[85]
- there is $\delta > 0$
- such that for all real x,

$$0 < x - a < \delta \quad \text{implies} \quad f(x) > M.$$

To get the definition of f approaching the limit minus infinity near a from above ($\lim_{x\to a^+} f(x) = -\infty$), change "$M > M_0$" and "$f(x) > M$" above to "$M < M_0$" and "$f(x) < M$".

To get the definition of f approaching the limit plus infinity near a *from below*, or *from the left* ($\lim_{x\to a^-} f(x) = (+)\infty$), change "$0 < x - a < \delta$" above to "$-\delta < x - a < 0$".

To get the definition of f approaching the limit minus infinity near a from below ($\lim_{x\to a^-} f(x) = -\infty$), change "$M > M_0$", "$f(x) > M$" and "$0 < x - a < \delta$" above to "$M < M_0$", "$f(x) < M$" and "$-\delta < x - a < 0$".

As an example, we verify formally the intuitively clear result that

$$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

Given $M < 0$, we seek $\delta > 0$ such that $x \in (1-\delta, 1)$ implies $1/(x-1) < M$. Now for $x < 1$ we have $x - 1 < 0$, so in this range $1/(x-1) < M$ is equivalent to $1 > M(x-1)$, and for $M < 0$ this is in turn equivalent to $1/M < x - 1$, or $x > 1 + 1/M$. From this it is clear that if we take $\delta = -1/M$ (note that this is positive, since we are assuming $M < 0$[86]), then $x \in (1-\delta, 1)$ indeed implies $1/(x-1) < M$.

[85] As observed after the definition of an infinite limit, this M_0 can be completely arbitrary.

[86] Without this (valid) assumption, the limit calculation would be rather more awkward.

As well as infinite limits, a very natural notion that slightly generalizes our concept of a limit is that of a "limit at infinity", capturing the behavior of a function as the input grows unboundedly large in magnitude, either positively or negatively.

Definition of a function approaching a limit at infinity Suppose that f is *defined near infinity* (or *near plus infinity*), meaning that there is some real number M such that f is defined at every point in the interval (M, ∞). Say that f *approaches the limit L near infinity* (or *near plus infinity*), denoted

$$\lim_{x \to \infty} f(x) = L,$$

if

- for all $\varepsilon > 0$
- there is a real number M
- such that for all x,

$$x > M \text{ implies } |f(x) - L| < \varepsilon.$$

Formulating precise definitions of

- $\lim_{x \to -\infty} = L$
- $\lim_{x \to \infty} = \infty$
- $\lim_{x \to \infty} = -\infty$
- $\lim_{x \to -\infty} = \infty$ and
- $\lim_{x \to -\infty} = -\infty$

are left as an exercise.

Here's an example. We claim that $\lim_{x \to \infty} \frac{x}{x+1} = 1$. To prove this entails showing that for all $\varepsilon > 0$ there is an M such that $x > M$ implies $x/(x+1) \in (1-\varepsilon, 1+\varepsilon)$. Let us initially commit to choosing that $M \geq -1$, so that for $x > M$ we have $x + 1 > 0$, and we do not run into any issues with attempting to divide by 0.

Now for all x we have $x < x + 1$, and so for those x satisfying $x + 1 > 0$ we have $x/(x+1) < 1$; so our goal is to ensure $x/(x+1) > 1 - \varepsilon$. But (again remembering that $1 + x > 0$, and also using $\varepsilon > 0$) we have that $x/(x+1) > 1 - \varepsilon$ is equivalent to $x > (1/\varepsilon) - 1$. So if we take M to be anything that is at least as large as both -1 and $(1/\varepsilon) - 1$, for example, $M = \max\{-1, (1/\varepsilon) - 1\}$, then $x > M$ implies $x/(x+1) \in (1-\varepsilon, 1+\varepsilon)$, as required[87].

Here are some general facts about limits at infinity, all of which you should be able to prove, as the proofs are very similar to related statements about ordinary limits (limits near a finite number).

[87] In fact, for $\varepsilon > 0$ it holds that $(1/\varepsilon) - 1 > -1$, so we could have simple said "take $M = (1/\varepsilon) - 1$".

Theorem 6.7.

- *If* $\lim_{x\to\infty} f(x) = L$ *and* $\lim_{x\to\infty} g(x) = M$*, then*
  - $\lim_{x\to\infty}(f+g)(x) = L+M$*;*
  - $\lim_{x\to\infty}(fg)(x) = LM$*;*
  - $\lim_{x\to\infty} cf(x) = cL$*; and*
  - $\lim_{x\to\infty}(f/g)(x) = L/M$*, provided* $M \neq 0$*.*
- *For* $n \in \mathbb{N} \cup \{0\}$*,*
  - $\lim_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \end{cases}$ *and*
  - $\lim_{x\to\infty} \frac{1}{x^n} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$
- *Suppose* $p(x)$ *is the polynomial* $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$*, and* $q(x)$ *is the polynomial* $q(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0$[88] *(*$n, m \geq 0$*). Then*

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 1 & \text{if } n = m \\ \infty & \text{if } n > m \\ 0 & \text{if } n < m. \end{cases}$$

Proof: We'll just prove two of the statements above, leaving the rest as exercises. First, suppose $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$. We will consider $\lim_{x\to\infty}(fg)(x)$, and show that it equals LM. We have

$$\begin{aligned} |(fg)(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &\leq |f(x) - L||g(x)| + |L||g(x) - M|. \end{aligned}$$

Since $\lim_{x\to\infty} g(x) = M$ we know that there is X_1[89] such that $x > X_1$ implies $g(x) \in (M-1, M+1)$, so $|g(x)| \leq |M|+1$. Now let $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = L$ we know that there is X_2 such that $x > X_2$ implies $|f(x) - L| < \varepsilon/(|M|+1)$. Since $\lim_{x\to\infty} g(x) = M$ we know that there is X_3 such that $x > X_3$ implies $|g(x) - M| < \varepsilon/(|L|+1)$[90]. It follows that if $x > \max\{X_1, X_2, X_3\}$ then

$$\begin{aligned} |(fg)(x) - LM| &\leq |f(x) - L||g(x)| + |L||g(x) - M| \\ &\leq |f(x) - L|(|M|+1) + (|L|+1)|g(x) - M| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

[88] This corollary of the previous parts could have been formulated for more general polynomials, with arbitrary (positive of negative) leading coefficients; but the statement would be messy, and in any case by pulling out an appropriate constant, the ratio of two arbitrary polynomials can always be reduced to the form presented above.

[89] We have to change notation slightly from the definition, since M is now being used for something else.

[90] We bound by $\varepsilon/(|L|+1)$ here, rather than $\varepsilon/|L|$, to avoid the possibility of dividing by 0

so $\lim_{x\to\infty}(fg)(x) = LM$, as claimed.

Let's also prove $\lim_{x\to\infty} x^n = \infty$ if $n > 0$. We haven't formulated the relevant definition, but of course what this must mean is that for all M (and, if we wish, we can take this M to be positive, or bigger than any fixed constant M_0) there is an N such that $x > N$ implies $x^n > M$.

Let's commit to only considering $M \geq 1$. If we take $N = M$, then $x > N$ implies $x > M$, which in turn implies (because $M \geq 1$) that $x^n > M$, and we have the required limit. □

Returning to the previous example, $\lim_{x\to\infty} \frac{x}{x+1}$: that the limit exists and is 1 follows easily, from the above theorem. Formulating an analogous result for limits near minus infinity is left as an exercise.[91]

[91] For plenty of exercises on the kinds of limits introduced in this section, see Spivak, Chapter 5, questions 32-41.