Math 30210 — Introduction to Operations Research

Examination 1 (15 points per question, 90 points total)

Solutions

1. (a) A linear programming problem calls for the minimization of $x + 2y - 4w$. The problem has two artificial variables, R_1 and R_2. Which of the following is the correct initial objective function, if the problem is being solved using the M-method?

   - $x + 2y - 4w + MR_1 + MR_2$ with $M > 0$ a large number
   - $x + 2y - 4w - MR_1 - MR_2$ with $M > 0$ a large number

   Justify your choice.

   Solution: The first of the two is the correct choice. Having a large multiple of either of R_1, R_2 added to the objective causes it to increase. Since the problem is a minimization, this provides an incentive to have R_1, R_2 zero at the optimum, and a penalty to having either of them positive.

   (b) You are solving a minimization problem using the simplex method. How can you tell when you have reached the optimal tableau?

   Solution: If there is any variable whose coefficient in the objective row is positive, the optimum has not yet been reached, as that variable may be brought into the set of basic variables to reduce the objective. If all the variables have coefficient zero or less in the objective row, no new variable may be brought in to lower the objective, and the optimum has been reached.

   (c) You are solving a linear programming problem using the two-phase method. How can you tell whether there is a feasible solution to the problem?

   Solution: If phase 1 ends by showing that the minimum of the sum of the artificial variables is zero, there is a feasible solution to the problem; otherwise, there is no feasible solution.

   (d) At the end of each iteration of the simplex method, the objective function is written exclusively in terms of non-basic variables. Why is this important for the running of the algorithm?

   Solution: Expressing the objective in terms of non-basic variables makes it obvious which of the non-basic variables can be introduced into the set of basic variables so as to improve the objective. More precisely, bringing a non-basic variable into the set of basic variables causes all of the basic variables to change value. Some may increase and some may decrease. So if even a single basic variable had a non-zero coefficient in the objective, it would be impossible to cleanly keep track of how the objective changes as a non-basic variable is introduced. But with the objective expressed in terms of non-basic (i.e., currently 0) variables, it is clear that in a maximization problem, introducing a non-basic variable with a negative coefficient improves the objective, and introducing one with a positive coefficient hurts the objective; and vice-versa for a minimization problem.

2. Due to an unexpected glut of orders, Blaster Steel has decided to hire temporary workers for a four day period. Each temp can work either two consecutive days (for total pay of $250), or three consecutive days (for total pay of $300). At least 10 temp workers are needed on days 1 and 3, and at least 15 on days 2 and 4. New three-day workers cannot be hired on days 3 or 4, and no new worker can be hired on day 4.

   (a) Mathematically formulate the problem of hiring temporary workers to satisfy Blaster Steel's needs, while minimizing cost. You should clearly identify your variables, the objective function, and the constraints.

   Solution: Let x_1, x_2 and x_3 be the number of two-day workers hired on days 1, 2 and 3, respectively, and y_1, y_2 be the number of three-day workers hired on days 1 and 2, respectively.
   Minimize $250(x_1 + x_2 + x_3) + 300(y_1 + y_2)$ subject to

$$
\begin{aligned}
x_1 + y_1 &\geq 10 \\
x_1 + x_2 + y_1 + y_2 &\geq 15 \\
x_2 + x_3 + y_1 + y_2 &\geq 10 \\
x_3 + y_2 &\geq 15
\end{aligned}
$$

   with all x_i, $y_j \geq 0$, and all x_i, y_j integers.

   (b) Due to limited availability of training personnel, at most 10 new workers can start their shifts on any day. Add constraints to your model to take this into account.

   Solution: Add $x_1 + y_1 \leq 10$, $x_2 + y_2 \leq 10$ and $x_3 \leq 10$

   (c) Union regulations demand that at least half of all money that is used for temp worker pay be used for workers on three-day shifts. Add a constraint to your model to take this into account.

   Solution: $300(y_1 + y_2) \geq 250(x_1 + x_2 + x_3)$ (or

$$300(y_1 + y_2) \geq \frac{1}{2}(250(x_1 + x_2 + x_3) + 300(y_1 + y_2)))$$

3. Put the following linear programming problem into standard form.

   Maximize $3x + 5y - 2z$ subject to the constraints

$$\begin{aligned} x + y &= -5 \\ 2x - 4z &\leq 2 \\ y - z &\geq 2 \end{aligned}$$

   with $x \geq 0$, y unrestricted and $z \leq 0$.

   Solution: We multiple the first constraint by -1 to make right hand side positive; add a slack variable s_1 to the second constraint and subtract a surplus variable s_2 from the third to make them both equalities, replace y with $y^- - y^+$ and z with $z^- - z^+$, add a constraint $z^- - z^+ \leq 0$, and add a slack variable s_3 to this new constraint to make it an equality.

   Maximize $3x + 5y^- - 5y^+ - 2z^- + 2z^+$ subject to the constraints

$$\begin{aligned} -x - y^- + y^+ &= 5 \\ 2x - 4z^- + 4z^+ + s_1 &= 2 \\ y^- - y^+ - z^- + z^+ - s_2 &= 2 \\ z^- - z^+ + s_3 &= 0 \end{aligned}$$

   with all variables ≥ 0.

   Alternatively: Since $z \leq 0$, we could make a change of variables: $w = -z$; automatically we must have $w \geq 0$. The problem then becomes:

   Maximize $3x + 5y^- - 5y^+ + 2w$ subject to the constraints

$$\begin{aligned} -x - y^- + y^+ &= 5 \\ 2x + 4w + s_1 &= 2 \\ y^- - y^+ + w - s_2 &= 2 \end{aligned}$$

   with all variables ≥ 0.

4. Below are three simplex tableau encountered while solving linear programming problems. In each case identify

   - the current basic variables, their values, and the objective value
   - the entering variable and the departing variable, if any
   - the pivot element of the tableau, if any (circle it).

   If there are no entering and departing variables and/or pivot elements, say why.

(a)

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln.
Max	1	0	0	-6	3	-2	0	4	46
x_7	0	0	0	4	3	4	2	1	7
x_1	0	1	0	**2**	1	-2	2	0	3
x_2	0	0	1	-1	0	-2	2	0	2

Solution:
Current basic variables (and values): $x_7 = 7$, $x_1 = 3$, $x_2 = 2$

Current objective value: $z = 46$

Entering and departing variable: x_3 and x_1, respectively

Pivot entry bolded above

(b)

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln.
Min	1	0	-1	-2	-3	-4	0	0	16
x_1	0	1	-2	1	2	2	0	0	2
x_7	0	0	-3	1	5	3	0	1	1
x_6	0	0	1	-1	4	1	1	0	2

Solution:
Current basic variables (and values): $x_1 = 2$, $x_7 = 1$, $x_6 = 2$

Current objective value: $z = 16$

No entering variable, departing variable or pivot entry: optimum has been reached (there are no positive coefficients along the objective row)

(c)

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln.
Min	1	4	-1	0	0	0	-2	0	8
x_5	0	-1	2	2	0	1	4	0	3
x_7	0	-1	3	-6	0	0	-5	1	2
x_4	0	-3	3	5	1	0	3	0	6

Solution:
Current basic variables (and values): $x_5 = 3$, $x_7 = 2$, $x_4 = 6$

Current objective value: $z = 8$

Entering and departing variable: x_1 is entering variable. No departing variable of pivot entry: all ratios of entries in pivot column to corresponding entries in solution column are negative, so x_1 can enter at arbitrarily high values. The solution space, and optimum, are unbounded

5. Set up the initial simplex tableau for the following linear programming problem, and then modify the objective row so that the simplex algorithm can be started. You may use either the M-method or the two-phase method, but say clearly which one you

are using. You can use the templates below to create your tableaus. (Since it's only the z-row that gets modified, I have X-ed out the constraint rows of the modified tableau).

Maximize $3x + 5y - 2w$ subject to the constraints

$$\begin{aligned} x + y + w &= 5 \\ 2x - y + 4w &\geq 2 \\ 4x - y - w &\geq 2 \end{aligned}$$

with $x, y, w \geq 0$.

Solution: (Using M-method, with $M = 100$)

We begin by adding surplus variables s_1, s_2 to constraints 2 and 3, and artificial variables R_1, R_2, R_3 to constraints 1, 2 and 3, and modifying the objective to maximize $3x + 5y - 2w - 100R_1 - 100R_2 - 100R_3$.

(a) Initial tableau

Basic	z	x	y	w	s_1	s_2	R_1	R_2	R_3	Soln.
Max	1	-3	-5	2	0	0	100	100	100	0
R_1	0	1	1	1	0	0	1	0	0	5
R_2	0	2	-1	4	-1	0	0	1	0	2
R_3	0	4	-1	-1	0	-1	0	0	1	2

(b) Tableau with modified z-row (objective expressed in terms of non basic variables)

Basic	z	x	y	w	s_1	s_2	R_1	R_2	R_3	Soln.
Max	1	-703	95	-398	100	100	0	0	0	-900
X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X

Solution: (Using Two-phase method)

We begin by adding surplus variables s_1, s_2 to constraints 2 and 3, and artificial variables R_1, R_2, R_3 to constraints 1, 2 and 3, and modifying the objective to minimize $R_1 + R_2 + R_3$.

(a) Initial tableau

Basic	z	x	y	w	s_1	s_2	R_1	R_2	R_3	Soln.
Min	1	0	0	0	0	0	-1	-1	-1	0
R_1	0	1	1	1	0	0	1	0	0	5
R_2	0	2	-1	4	-1	0	0	1	0	2
R_3	0	4	-1	-1	0	-1	0	0	1	2

(b) Tableau with modified z-row (objective expressed in terms of non basic variables)

Basic	z	x	y	w	s_1	s_2	R_1	R_2	R_3	Soln.
Max	1	7	-1	4	-1	-1	0	0	0	9
X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X

6. Consider the linear program

Maximize $5x + 3y + w$ subject to the constraints

$$\begin{aligned} x + y &\le 6 \\ 5x + 3y + 6w &\le 15 \end{aligned}$$

with $x, y, w \ge 0$, and the following optimal simplex tableau that occurs during the solving of this problem:

Basic	z	x	y	w	s_1	s_2	Soln.
Max	1	0	0	5	0	1	15
s_1	0	0	.4	-1.2	1	-.2	3
x	0	1	.6	1.2	0	.2	3

(a) Read off an optimal solution to the problem from this tableau

Solution: $x = 3$, $y = w = 0$ (and $z = 15$)

(b) Find an alternative optimal solution. (You may use the tableau below if you wish).

Basic	z	x	y	w	s_1	s_2	Soln.
Max	1	0	0	5	0	1	15
s_1	0	$-2/3$	0	-2	1	$-1/3$	1
y	0	$5/3$	1	2	0	$1/3$	5

Solution: y is non-basic, but has 0 coefficient in the objective row, so it can be brought into the set of basic variables to create a new optimal basic feasible solution. The departing variable should be x. The tableau above shows the result of pivoting on the y-x pivot entry. From it we read off a second optimal solution: $x = w = 0$, $y = 5$.

(c) Find a third optimal solution that is different from the two previously found.

Solution: Any vector (x, y, w) of the form $\lambda(3, 0, 0) + (1 - \lambda)(0, 5, 0)$, with $0 \le \lambda \le 1$ (i.e., any point on the line segment connecting the two alternative basic feasible solutions found above) is also an optimal solution; for example, at $\lambda = 1/2$ we get the point $x = 3/2$, $y = 5/2$, $z = 0$.