Math 30210 — Introduction to Operations Research

Examination 2 (50 points total)

Solutions

1. (26 pt total) Consider the following linear programming problem:

   Maximize
   $$3x_1 + 2x_2 + 5x_3$$
   subject to
   $$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & \leq & 430 \\ 3x_1 & + & & & 2x_3 & \leq & 460 \\ x_1 & + & 4x_2 & & & \leq & 420 \end{array}$$
   and $x_1, x_2, x_3 \geq 0$.

   Slack variables x_4, x_5 and x_6 are introduced to the three constraints, and the resulting optimal simplex tableau is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	soln
Max	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

   (a) (2 pt) What is the inverse matrix in the above tableau?

   $$\textbf{Solution}\text{: Inverse} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

   (b) (4 pt) What does the solution column of the final tableau change to if the right hand side of the first constraint changes from 430 to $430+D$? From this deduce the dual price of the first constraint.

   Solution: Solution column changes to
   $$\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 430 + D \\ 460 \\ 420 \end{bmatrix} = \begin{bmatrix} 100 + \frac{D}{2} \\ 230 \\ 20 - 2D \end{bmatrix}.$$

From this we infer that the optimum solution changes to $x_2 = 100 + \frac{D}{2}, x_3 = 230$ (as long as all three quantities in the solution column remain non-negative), and so the optimum changes to $1350 + D$. From this we can deduce that the dual price of the first constraint is 1.

(c) (3 pt) What is the range of values of the right-hand side of the first constraint for which the dual price is valid?

Solution: The dual price remains valid as long as $100 + \frac{D}{2}$, 230 and $20 - 2D$ all remain positive; that is, as long as $-200 \leq D \leq 10$, leading to a valid range of $(230, 440)$ for the dual price.

(d) (2 pt) Write down the dual problem.

Solution: Minimize

$$430y_1 + 460y_2 + 420y_3$$

subject to

$$\begin{array}{rcrcrcr} y_1 & + & 3y_2 & + & y_3 & \geq & 3 \\ 2y_1 & + & & & 4y_3 & \geq & 2 \\ y_1 & + & 2y_2 & & & \geq & 5 \end{array}$$

and $y_1, y_2, y_3 \geq 0$.

(e) (4 pt) Use the inverse matrix to determine the optimum solution to the dual problem (the values of the dual variables corresponding to the current tableau).

Solution:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}.$$

So the dual solution is $y_1 = 1, y_2 = 2$ and $y_3 = 0$.

(f) (3 pt) Suppose that the coefficient of x_1 in the objective changes from 3 to P. Using the values of the dual variables computed in the last part, determine what the objective row coefficient of x_1 changes to in the optimal tableau, and say for what values of P this coefficient is negative.

Solution: After the change, the dual constraint corresponding to x_1 is

$$y_1 + 3y_2 + y_3 \geq P.$$

The right-hand side of this is P, and the left-hand side, evaluated at the current dual solution, is 7, so the objective row coefficient of x_1 changes to $7 - P$. (Notice that this is 4 when $P = 3$). This coefficient is negative as long as $P > 7$.

(g) (3 pt) Suppose that x_1, x_2 and x_3 represent amounts of three different goods, say A, B and C, being produced by a factory. The production process involves three machines, I, II and III. The first constraint encodes the fact that units of goods A, B and C take up 1, 2 and 1 minutes respectively on machine I, and that there are 430 minutes a day of machine I time available. The remaining constraints similarly refer to machines II and III. Profit per unit of goods A, B and C is 3, 2 and 5 dollars, respectively.

Interpret your answer to the last part of this question.

Solution: If the coefficient of x_1 in the objective row remains positive after the change, then the current tableau remains optimal, and the optimal production mix continues to call for the factory to produce 100 units of B and 230 of C, but none of A. If the coefficient becomes negative, the current tableau becomes non-optimal, and the simplex algorithm calls for introducing x_1 as a basic variable, leading to an optimal production mix in which some non-zero number of units of A are produced.

The economic interpretation of the last part is therefore that the production of A only becomes profitable if the profit per unit of A is greater than \$7.

2. (8 pt total) Consider the following linear programming problem:

Minimize

$$x_1 + 2x_2 + 3x_3 + 4x_4$$

subject to

$$4x_1 + 3x_2 + 2x_3 + x_4 \geq 1$$

and $x_1, x_2, x_3, x_4 \geq 0$.

(a) (3 pt) Write down the dual problem.

Solution: Maximize y subject to

$$\begin{aligned} 4y &\leq 1 \\ 3y &\leq 2 \\ 2y &\leq 3 \\ y &\leq 4 \end{aligned}$$

and $y \geq 0$.

(b) (3 pt) Solve the dual problem by inspection.

Solution: The strongest of the four constraints on y is the first one, which says $y \leq \frac{1}{4}$. So the linear programming problem can be reduced to:

$$\text{Maximize } y \text{ subject to } 0 \leq y \leq \tfrac{1}{4}.$$

The optimum is clearly $\frac{1}{4}$, achieved at $y = \frac{1}{4}$.

(c) (2 pt) What is the optimum value of the primal objective function?

Solution: The optimum is $\frac{1}{4}$, since by the strong duality theorem the optimum of the primal is the same as the optimum of the dual.

3. (9 pt total) I have a linear programming minimization problem to solve. I find an initial basic solution, and I express the objective only in terms of non-basic variables. I set up an initial simplex tableau. I try to decide whether I should proceed to solve the problem using the primal simplex algorithm, the dual simplex algorithm or the generalized simplex algorithm. Help me decide! In your answer, you should tell me how I should decide based on looking at the numbers in the simplex tableau alone (you should not use phrases like "current solution is optimal/better than optimal/feasible", etc.).

   (a) (3 pt) Under what circumstances should I use the primal simplex algorithm?

   Solution: If all of the entries in the solution column are positive, and some of the entries in the objective row are positive.

   (b) (3 pt) Under what circumstances should I use the dual simplex algorithm?

   Solution: If some of the entries in the solution column are negative, and all of the entries in the objective row are negative.

   (c) (3 pt) Under what circumstances should I use the generalized simplex algorithm?

   Solution: If some of the entries in the solution column are negative, and some of the entries in the objective row are positive.

4. (12 pt total) SantaCo produces Christmas trees at two locations, one in Vermont and one in Oregon. They distribute to three states, Indiana, Kansas and Nebraska. This year's production capacity in Oregon (measured in thousands of trees) is 40, and in Vermont is 30. The demands in IN, KS and NE are 20, 25 and 35 thousand trees, respectively. Shipping costs from Oregon to IN, KS and NE (measured in thousands of dollars per thousand tress) are 7, 6 and 8, respectively. Shipping costs from Vermont to IN, KS and NE are 5, 7 and 9, respectively. No shortfall of demand will be tolerated in Indiana, and for each thousand tree shortfall in KS and NE, extra costs of 2 and 3 thousand dollars, respectively, are incurred.

   (a) (4 pt) Using a dummy location, set this problem up as a transportation problem in the tableau below.

Solution:

	IN	KS	NE	supply
Oregon	7	6	8	40
Vermont	5	7	9	30
Dummy	100	2	3	10
demand	20	25	35	

Here we have chosen $M = 100$ to prevent any trees from being shipped from the Dummy source to IN (i.e., to prevent shortfall at IN).

(b) (3 pt) Use the least-cost method to generate an initial basic feasible solution.

Solution:

	IN	KS	NE	supply
Oregon	7	6 15	8 25	40
Vermont	5 20	7	9 10	30
Dummy	100	2 10	3	10
demand	20	25	35	

(c) (5 pt) Use the method of multiplies to determine whether the least-cost method gives the optimal solution. Justify your answer.

Solution:

	$v_1 = 4$ IN	$v_2 = 6$ KS	$v_3 = 8$ NE	supply
$u_1 = 0$ Oregon	7 (-3)	6 15	8 25	40
$u_2 = 1$ Vermont	5 20	7 (0)	9 10	30
$u_3 = -4$ Dummy	100 (-100)	2 10	3 (1)	10
demand	20	25	35	

Setting $u_1 = 0$, the remaining multipliers are found by solving $u_i + v_j = c_{ij}$ for each basic x_{ij}. The figures in brackets in the non-basic cells are the non-basic objective coefficients $u_i + v_j - c_{ij}$. Since one of these (in the Dummy-NE cell) is positive, the optimum has not yet been reached.

(d) **Bonus:** (2pt) Find the optimum solution to this transportation problem.

Solution:

	IN	KS	NE	supply
Oregon	7	6 25	8 15	40
Vermont	5 20	7	9 10	30
Dummy	100	2	3 10	10
demand	20	25	35	

(This is the tableau obtained from the least-cost solution after one iteration; it turns out to be optimal.)