Math 30210 — Introduction to Operations Research

Assignment 1 (50 points total)

Due before class, Wednesday September 5, 2007

Instructions: Please present your answers neatly and legibly. Include a cover page with your name, the course number, the assignment number and the due date. The course grader reserves the right to leave ungraded any assignment that is disorganized, untidy or incoherent. You may turn this assignment in before class, or leave it in my mailbox (outside 255 Hurley Hall). It can also be emailed; if you plan to email, please check with me to see if the format you plan to use is one that I can read. No late assignments will be accepted. It is permissible (and encouraged) to discuss the assignments with your colleagues; but the writing of each assignment must be done on your own.

Reading: Chapter 1, and Sections 2.1, 2.2 and 2.3

1. (2 points) Taha, Problem Set 1.1A, problem 1.

   Solution: Many possible solutions; for example (a really silly one) one could buy a total of ten one-way tickets, five from FYV to DEN to be used on Mondays, and five from DEN to FYV to be used on Wednesdays.

2. (2 points) Addition to the previous problem: identify a feasible alternative (other than Alternative 3) that is also optimal.

   Solution: More than one possible solution; for example one could buy a FYV-DEN-FYV round trip to be used on the first Monday and last Wednesday, a DEN-FYV-DEN round trip to be used on the first Wednesday and last Monday, a FYV-DEN-FYV round trip to be used on the second Monday and second from last Wednesday, and two DEN-FYV-DEN round trips, one for the second Wednesday and third Monday and one for the third Wednesday and fourth Monday. (All trips are round trips, and all span a Saturday).

3. (5 points) Taha, Problem Set 1.1A, problem 4.

   Solution: a) E.g.: Amy and Jim cross, Jim returns, Jim and John cross, John returns, John and Kelly cross ($2+2+5+5+10 = 24$ minutes); or, Amy and Jim cross, Amy returns, Amy and John cross, Amy returns, Amy and Kelly cross ($2+1+5+1+10 = 19$ minutes).

b) I'm not sure what this question means. I presume it's as simple as: we evaluate an alternatives by looking at the sum of the times of the individual journeys within the alternative; which ever alternative has the lowest total is the best.

c) The following alternative takes 17 minutes, and is the optimal: Amy and Jim cross, Amy returns, John and Kelly cross, Jim returns, Amy an Jim cross ($2+1+10+2+2 = 17$ minutes). (For proof of optimality, see the next question).

4. (17 points total) A more general version of the previous problem:

   Four friends are gathered on one side of a river. (Their names are Stuhldreher, Miller, Crowley and Layden, but we will call them F_1, F_2, F_3 and F_4.)

   They want to cross to the other side of the river, but they only have one rowboat which can carry a maximum of two people at one time. F_1 can row across the river in a_1 minutes, F_2 in a_2 minutes, etc. For the sake of convenience, we has listed the friends in such a way that $a_1 \leq a_2 \leq a_3 \leq a_4$.

   If two people are in the boat, the time taken to cross is that of the slower of the two rowers (e.g., if F_1 and F_2 row together, the journey will take a_2 minutes).

   The rowboat cannot cross the river without a rower in it; also, it is an old rowboat, and can only manage a total of five one-way journeys before it sinks.

   (a) (2 points) How many feasible schemes are there to get the friends across the river?

   Solution: There are 108 possible feasible plans that involve the minimum possible number of trips across the river (five). There are 6 choices for the first pair to cross, 2 choices for the first person to return, 3 choices for the next pair to cross, and 3 choices for the second person to return; after that there is no more choice. Total: 6x2x3x3 $= 108$.

   (b) (2 points) Describe a scheme which you suspect minimizes the time taken for the friends to cross the river. How long does it take?

   Solution: My original suspicion was the following: F_1 and F_2 cross, F_1 returns, F_1 and F_3 cross, F_1 returns, F_1 and F_4 cross (i.e., F_1 acts as a shuttler). Total time: $a_2 + a_1 + a_3 + a_1 + a_4 = 2a_1 + a_2 + a_3 + a_4$.

   (c) (7 points) Prove that the scheme you have described is indeed the best. (In the course of answering this part, you may discover that your originally proposed scheme is *not* the best, or is only the best for certain values of a_1, a_2, a_3 and a_4, in which case you should start over...)

   Solution: A systematic examination of the 108 feasible alternatives described in the first part can be performed by forming a tree that starts from a single root and initially has 6 branches (one for each of the 6 initial choices), with each branch having 2 subbranches (one for each of the 2 second choices), each subbranch having 3 subsubbranches (one for each of the 3 third choices), and each subsubbranch having 3 leaves (one for each of the 3 fourth choices). There are 108 leaves, and 108 paths from root to leaves, one for each feasible alternative. Each branch (and subbranch, etc) represents a single journey across the river of a computable time, and each leave represents a pair of journeys across the river of a computable time; so each branch, etc., can be labeled with a time in such a way that the total time of an alternative can be computed by simply summing the times along the branches and leaf of the path corresponding to that alternative (see Figure 1).

Performing this examination, we find that there are a total of 6 alternatives that take a total of $2a_1 + a_2 + a_3 + a_4$ minutes (in all of these, F_1 acts as shuttler; the 6 comes from the fact that there are 3 ways to choose who is dropped off first, 2 ways to choose who is dropped off second, 1 way to choose who is dropped off third, and $6 =$ 3x2x1). There are two alternative that takes $a_1 + 3a_2 + a_4$ minutes (F_1 and F_2, F_1, F_3 and F_4, F_2, F_1 and F_2 is one; F_1 and F_2, F_2, F_3 and F_4, F_1, F_1 and F_2 is the other). Any other alternative is beaten by at least one of these two. But which of these two is better? It turns out to depend on the specific values of a_1 through a_4. Specifically, the first alternative is better if

$$2a_1 + a_2 + a_3 + a_4 < a_1 + 3a_2 + a_4,$$

that is, if $a_1 + a_3 < 2a_2$. The second is better if $a_1 + a_3 > 2a_2$. If $a_1 + a_3 = 2a_2$, the two alternatives are equally good. (In the case of Amy and her friends in the previous question, we have $1 + 10 >$ 2x5, so the second alternative is the best; I had originally thought that the first was the best).

For the particular objective in this problem, we didn't really have to look at all 108 feasible schemes. We can take the following shortcuts: essentially, the first thing that has to happen is that one of the four friends has to be dropped on the opposite bank. If it is to be F_1, then clearly F_2 should be used as a shuttler (for a time of $2a_2$). If it is to be F_2, F_3 or F_4, then clearly F_1 should be used as a shuttler (for times of $a_1 + a_2$, $a_1 + a_3$ or $a_1 + a_4$). For each of these four options, there are three ways to continue: one person has to be left on the near bank, while two go on to the opposite bank. But from there, there is only one sensible way to proceed: the fastest available rower on the opposite bank goes back to the near back to pick up the last remaining person. In this way we see that there are really only *twelve* feasible schemes worth considering, giving the objective we are trying to minimize. These are summarized below (with associated times shown in parentheses):

F_1F_2	F_2	F_2F_3	F_1	F_1F_4	$(a_1 + 2a_2 + a_3 + a_4)$
F_1F_2	F_2	F_2F_4	F_1	F_1F_3	$(a_1 + 2a_2 + a_3 + a_4)$
F_1F_2	F_2	F_3F_4	F_1	F_1F_2	$(a_1 + 3a_2 + a_4)$
F_1F_2	F_1	F_1F_3	F_1	F_1F_4	$(2a_1 + a_2 + a_3 + a_4)$
F_1F_2	F_1	F_1F_4	F_1	F_1F_3	$(2a_1 + a_2 + a_3 + a_4)$
F_1F_2	F_1	F_3F_4	F_2	F_1F_2	$(a_1 + 3a_2 + a_4)$
F_1F_3	F_1	F_1F_2	F_1	F_1F_4	$(2a_1 + a_2 + a_3 + a_4)$
F_1F_3	F_1	F_1F_4	F_1	F_1F_2	$(2a_1 + a_2 + a_3 + a_4)$
F_1F_3	F_1	F_2F_4	F_2	F_1F_2	$(a_1 + 2a_2 + a_3 + a_4)$
F_1F_4	F_1	F_1F_2	F_1	F_1F_3	$(2a_1 + a_2 + a_3 + a_4)$
F_1F_4	F_1	F_1F_3	F_1	F_1F_2	$(2a_1 + a_2 + a_3 + a_4)$
F_1F_4	F_1	F_2F_3	F_2	F_1F_2	$(a_1 + 2a_2 + a_3 + a_4)$

Clear any of the six schemes with time $2a_1 + a_2 + a_3 + a_4$ are better than any of the four schemes with time $a_1 + 2a_2 + a_3 + a_4$ (since $a_1 \leq a_2$), so we are very quickly lead to compare $2a_1 + a_2 + a_3 + a_4$ with $a_1 + 3a_2 + a_4$ as the two potentially best times.

(d) Critique the model presented in this problem; specifically:

i. (1 point) Do you think that this is a realistic model?
Solution: I don't think it's realistic; see below.

ii. (2 point) Are there any factors that model ignores?
Solution: I can think of many:

- Because of river currents, crossing times depend on direction of crossing.
- The more times a rower crosses the river, the more tired she becomes, so crossing times should increase as number of journeys made increases.
- It's reasonable that crossing time should increase if there are two people in the boat, but unreasonable that it should increase all the way up to the time of the slower of the two rowers — if the faster of the two rows, the journey should just be slowed down slightly by the increase in weight.
- More, I'm sure, that others have thought of.

iii. (3 points) Can you propose what you think might be a better model?
Solution: I propose: Each rower has base crossing times w (with current) and a (against current), a fatigue factor $f > 1$ and an extra-weight factor $e > 1$. Crossing time if alone and with current is

$$w f^{\#\text{ previous journeys made as rower}};$$

Crossing time if alone and against current is

$$af^{\#\text{ previous journeys made as rower}};$$

if two people are in the boat, the designated rower is the one whose crossing time for that journey would be faster, if the two were crossing on their own, and the crossing time is that rowers individual crossing time (computed by the previous formula) times e.

5. (Optional!) Taha, Problem Set 1.1A, problem 6.

6. (2 points) Consider the first case described in Taha, Section 1.5 (the case involving the elevators). Critique the proposed solution. Specifically, do you think it satisfactorily resolves the problem?

   Solution: I don't think it does! I imagine that people complained about the slow service because it was hurting their productivity; and the cosmetic solution, while distracting them from the realization of the delay, still leaves their productivity down. (BTW, I think that the company's HR team should launch an investigation into why they are hiring so many self-absorbed individuals!)

7. (Optional!) Taha, Problem Set 2.1A, problems 1-4.

   Note: for the next two items, you should set up the problem as a linear programming problem, assigning appropriate variables, identifying the objective function, and identifying all the constraints. Then you should solve the problem graphically. For the first problem (Taha 2.2A. problem 4) you **must** draw the graphs by hand. For the second and third, you may if you wish print out a TORA screenshot.

8. (10 points total) Taha, Problem Set 2.2A, problems 6 (5 points), 15 (5 points) and (Optional!) 16.

   Solution: 6) Let x be the number of sheets produced, and y the number of bars. We are told that sheets can be produced at a rate of 800 per day, so the time taken to produce x sheets is $x/800$. Similarly, the time taken to produce y bars is $y/600$. Since all this production must take place in the one day, we have the first (production) constraint:

   $$x/800 + y/600 \leq 1 \quad \text{or} \quad 3x + 4y \leq 2400.$$

   There are also two demand constraints:

   $$0 \leq x \leq 550, \quad 0 \leq y \leq 580.$$

   Subject to these constraints, we need to maximize the total profit

   $$40x + 35y.$$

(Note that the phrase “per ton” in the sentence that begins “The profit per ton is ...” seems to be an error). We solve this graphically (see Figure 2) and find that the optimum is $x = 550$, $y = 187.13$ (giving an objective value of 28,549.40 dollars). This is not really a feasible product mix, since we should probably produce a whole number of sheets. Rounding, the best mix seems to be 550 sheets and 187 bars (for a profit of 28,545 dollars).

15) Let x be the number of radio ads and y the number of TV ads purchased. Our task is to maximize

$$5000 + 2000(x - 1) + 4500 + 3000(y - 1) = 4500 + 2000x + 3000y$$

(note: I’m assuming here that no one listens to the radio **and** watches the TV!), subject to the constraints

$$300x + 2000y \leq 20000,$$

and

$$0 \leq 300x \leq 16000, \quad 0 \leq 2000y \leq 16000,$$

i.e.,

$$0 \leq x \leq 53\frac{1}{3}, \quad 0 \leq y \leq 8.$$

The graphical analysis (see Figure 3) suggests $x = 53\frac{1}{3}$, $y = 2$. Since one can’t purchase one third of an ad, I would propose a budget outlay of 15900 dollars for 53 radio ads and 4000 dollars for 2 TV ads, giving an objective value of 116,500.

9. (5 points total) Taha, Problem Set 2.2B, problems 4 (5 points) and (Optional!) 7.

   Solution: 4) Let x be number of hours worked at first store, and y the number of hours worked at the second store. Here our task is to minimize

$$8x + 6y$$

   subject to

$$x + y \geq 20, \quad 5 \leq x \leq 12, \quad 6 \leq y \leq 10.$$

   The graphical analysis (see Figure 4) shows that the optimum is achieved at $x = y = 10$, so John should split his time equally between the two stores (for a “stress index” of 140).

10. (7 points total) A furniture maker has 6 units of wood and 28 hours of free time, in which he will make decorative screens. Two models have sold well in the past, so he will restrict himself to those two. He estimates that model I requires 2 units of wood and 7 units of time, while model II requires 1 unit of wood and 8 hours of time. The prices for the models are \$ 120 and \$ 80 respectively. The furniture maker wishes to maximize his sales revenue.

(a) (2 points) Formulate this as a mathematical problem: assign appropriate variables, identify the objective function, and identify all the constraints.

Solution: Let x be the number of screens of model I that the furniture maker produces, and y the number of screens of model II. Our task is to maximize profit,

$$120x + 80y$$

subject to a time constraint and a materials constraint:

$$7x + 8y \leq 28, \quad 2x + y \leq 6,$$

two obvious non-negativity constraints, and two implicit integer constraints:

$$x, y \geq 0, \quad x, y \in \mathbb{N}.$$

(b) (1 point) Is the problem you formulated in the first part a linear programming problem of the type presented in Taha, Problem Sets 2.2A and 2.2B? If not, why not?

Solution: It is similar to the problems of 2.2A, since there was a integer constraint in those problems (you can't produce .6 of an aluminium sheet, buy half an ad, or sell $\frac{3}{4}$ of a decorative screen). Interestingly, Taha seems to ignore the integer constraints in 2.2A. It may be different to the problem of 2.2B, since there was not an integer constraint (John may perhaps be allowed to work $8\frac{1}{2}$ hours a week at one store, for example).

(c) (4 points) Solve the problem and identify how many screens of each model the furniture maker should make in order to maximize his profit.

Solution: Rather than using the graphical method to get a solution to the problem that may violate the integer constraint, and then do some rounding and "hope for the best", here I choose to roll up my sleeves and solve the integer problem directly. It is easy to see that the only pairs (x, y) that satisfy *all* the constraints (including the integer constraints) are $(0, 0)$, $(1, 0)$, $(2, 0)$, $(3, 0)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, $(0, 2)$, $(1, 2)$, and $(0, 3)$, leading respective to revenues of 0, 120, 240, 360, 80, 200, 320, 160, 280 and 240 dollars. The maximum revenue, 360 dollars, is achieved by taking $x = 3$ (three screens of model I) and $y = 0$ (no screens of model II). (Solving the problem graphically, ignoring the integer constraint, leads to a solution of $x = 2.22$, $y = 1.56$; it's far from clear how we would round this to directly to get the best integer solution. If we round by going to the nearest feasible integer point to $(2.22, 1.56)$, we would end up at $(2, 1)$, which leads to the suboptimal revenue of 320 dollars).