Converting an LP to standard form

All LP solvers first convert the given program to *standard form* which means

- all variables involved are restricted to be non-negative
- all constraints are equalities, with constant, non-negative right-hand sides

Converting may require new variables and rearranging constraints:

- an inequality can be multiplied by $-1$ to get non-negative rhs
- inequalities can be converted to equalities by adding or subtracting non-negative slack variables
- Unrestricted variables can be dealt with by writing the variable as the difference of two new non-negative variables
Example 1: the meatloaf problem

Recall the meatloaf problem, whose formulation was

Minimize

\[ 80x + 60y \]

subject to

\[ x + y \geq 1 \]
\[ -0.05x + 0.07y \leq 0 \]
\[ x, y \geq 0. \]

To convert to standard form, we introduce two new variables, \( s_1 \geq 0 \) and \( s_2 \geq 0 \). The first measures how much over 1 the quantity \( x + y \) is, and the second measures how much under 0 the quantity \( -0.05x + 0.07y \) is.
The meatloaf problem in standard form

Minimize

\[ 80x + 60y \]

subject to

\[ x + y - s_1 = 1 \]
\[ -.05x + .07y + s_2 = 0 \]
\[ x, y, s_1, s_2 \geq 0. \]

Note that if \((x, y, s_1, s_2)\) is feasible for this problem, then \((x, y)\) is feasible for the original; and if \((x, y)\) is feasible for the original, then \((x, y, (x + y) - 1, 0 - (-.05x + .07y))\) is feasible for this problem. Since the objective only involves \(x\) and \(y\), the two problems have the same solution.
Example 2: production without overtime

A company manufactures two products, A and B. The relevant production data is as follows

- Profit per unit: $2 and $5 respectively
- Labor time per unit: 2 hours and 1 hour respectively
- Machine time per unit: 1 hour and 2 hours respectively
- Available labor and machine time: 80 hours and 65 hours respectively

An easy linear program to maximize profit is

Maximize \(2x_A + 5x_B\)

subject to  
\[2x_A + x_B + s_1 = 80\]
\[x_A + 2x_B + s_2 = 65\]

where \(x_A, x_B \geq 0\) are amounts of A and B produced, respectively, and \(s_1, s_2 \geq 0\).
Example 3: production with overtime

Consider the same problem as before, but now with the wrinkle that labor and machine overtime may be purchased at a cost:

- Labor and machine overtime cost: $15 and $10 per hour, respectively

Now the labor constraint is

\[ 2x_A + x_B + s_1 = 80 \]

with \( s_1 \) unrestricted. It may possibly be positive (representing unused labor) or negative (representing overtime used). We have a similar unrestricted variable \( s_2 \) for the machine constraint, and the objective becomes the unwieldy (and non-linear)

\[ 2x_A + 5x_B \text{ (if } s_1, s_2 \text{ both positive) } \]
\[ 2x_A + 5x_B + 15s_1 \text{ (if } s_1 \text{ negative (so labor overtime used), } s_2 \text{ positive) } \]
\[ 2x_A + 5x_B + 10s_2 \text{ (if } s_1 \text{ positive, } s_2 \text{ negative) } \]
\[ 2x_A + 5x_B + 15s_1 + 10s_2 \text{ (if } s_1, s_2 \text{ both negative) } \]
Resolution

Write \( s_1 = s_1^- - s_1^+ \) and \( s_2 = s_2^- - s_2^+ \), all \( s_i^\pm \geq 0 \)

Interpretation:

\( s_1^- \) measures amount of unused labor
\( s_1^+ \) measures amount of overtime labor
\( s_1^- \) measures amount of unused machine time
\( s_1^+ \) measures amount of overtime on machines

The linear program in standard form:

Maximize \( 2x_A + 5x_B - 15s_1^+ - 10s_2^+ \) (a linear objective)

subject to \( 2x_A + x_B + s_1^- - s_1^+ = 80 \)
\( x_A + 2x_B + s_2^- - s_2^+ = 65 \)
where \( x_A, x_B, s_1^-, s_1^+, s_2^-, s_2^+ \geq 0 \).

Note there are feasible solutions with (say) \( s_1^-, s_1^+ > 0 \) (meaning unused labor and overtime). This is not realistic, but it is both intuitively and mathematically clear that this won’t occur in an optimal solution.