Going from graphic solutions to algebraic

- 2 variables:
  - Graph constraints
  - Identify corner points of feasible area
  - Find which corner point has best objective value

- More variables:
  - Think about constraints algebraically
  - Identify “extreme points” (a.k.a. basic feasible solutions)
  - Find which basic feasible solution has best objective value
Matrix representation

Optimize
\[ c_1 x_1 + \ldots + c_n x_n \]
subject to
\[ a_{11} x_1 + \ldots + a_{1n} x_n = b_1 \]
\[ \ldots \ldots \ldots \]
\[ a_{m1} x_1 + \ldots + a_{mn} x_n = b_m \]
and \( x_i \geq 0 \) for all \( i \), may be written as
Optimize
\[ c^T x \]
subject to
\[ Ax = b, \ x \geq 0 \]
for suitable vectors \( c, x, b \) and matrix \( A \)
Reducing to $m < n$

$Ax = b$ is a system of $m$ equations in $n$ unknowns

Linear algebra tells us that if $m > n$ there is redundancy in the system: some of the equations can be derived from linear combinations of others. Gaussian elimination allows us to reduce to a smallest possible set of independent equations with same solution space

If $m = n$ after reduction, the system either has no solution (in which case, no feasible solution to LP), or a unique solution, which may or may not be feasible (i.e., may or may not satisfy $x \geq 0$). In either case, the methods of linear algebra solve the LP

So from now on, we assume that $m < n$, meaning we have more variables than constraints. This is what we most often encounter in linear programming, since $\leq$ or $\geq$ constraints in the original problem introduce new variables, but no new constraints, and the same for unrestricted variables.
Basic feasible solutions

Take any $n - m$ of the $n$ variables and set them to zero. These variables are called the non-basic variables.

The matrix equation $Ax = b$ reduces to a system of $m$ equations in $m$ unknown variables. These variables are called the basic variables.

This $m$ by $m$ system may have a unique solution in which all the components are non-negative. If it does, the feasible solution to the LP with the non-basic variables set to zero, and the value of the basic variables determined by the unique solution to the $m$ by $m$ system, is called a basic feasible solution.

Just as the optimal solution to a 2-variable LP occurs at a corner point of the feasible space, the optimal solution to an LP in standard form always occurs at a basic feasible solution.
Solving an LP in finite time

- Convert to standard form, $m$ constraints, $n$ variables
- Use linear algebra to make sure there is no redundancy among equations (so $m \leq n$)
- If $m = n$, solve $Ax = b$ to perhaps get unique feasible solution
- If $m < n$:
  - For each subset of $n - m$ variables, set these variables to zero, and see if there is a unique, non-negative solution to the resulting $m$ by $m$ system. If there is, record this as a basic feasible solution
  - Evaluate the objective at all basic feasible solutions
  - The optimum is the best value among basic feasible solutions
Finite . . . but not necessarily quick!

How many basic feasible solutions are there? At most

\[ \binom{n}{m} = \frac{n!}{m!(n-m)!} \]

(this is the number of ways of choosing the \( m \) potentially basic variables from the set of \( n \) variables).

How long might the procedure just described take?

- \( n = 10, m = 5 \): 252 steps
- \( n = 20, m = 10 \): 184,756 steps
- \( n = 30, m = 15 \): 155,117,520 steps
- \( n = 40, m = 20 \): 137,846,528,820 steps
- \( n = 50, m = 25 \): 126,410,606,437,752 steps

Things quickly spiral out of control!
Why are basic feasible solutions so special?

Here’s a very high-level idea of what’s going on:

- If we set fewer than $n - m$ components of $x$ to zero, get to solve $m$ equations in more than $m$ variables to determine the rest of $x$. There are infinitely many solutions, and around any one solution there is a “ball” of other solutions. It’s always possible to find a direction to move in inside this ball that improves the objective value. Conclusion: no solution that is obtained by setting fewer than $n - m$ components to zero can be optimal

- If we set $n - m$ components of $x$ to zero, and the resulting $m$ by $m$ system has a solution, but not a unique one, then it has infinitely many, and we can do the same thing as in the first point to conclude that a solution so obtained isn’t optimal

- Conclusion: The only solutions that might potentially be optimal are the basic feasible solutions