Going from graphic solutions to algebraic

- 2 **variables**:
  - Graph constraints
  - Identify corner points of feasible area
  - Find which corner point has best objective value
- **More variables**:
  - Think about constraints algebraically
  - Identify “extreme points” (a.k.a. *basic feasible solutions*)
  - Find which basic feasible solution has best objective value

Matrix representation

Optimize

$$c_1 x_1 + \ldots + c_n x_n$$

subject to

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ \ldots &\ \ldots\ \ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

and $x_i \geq 0$ for all i, may be written as

Optimize

$$\mathbf{c}^T \mathbf{x}$$

subject to

$$A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$$

for suitable vectors $\mathbf{c}$, $\mathbf{x}$, $\mathbf{b}$ and matrix A

Reducing to $m < n$

$A\mathbf{x} = \mathbf{b}$ is a system of m equations in n unknowns

Linear algebra tells us that if $m > n$ there is redundancy in the system: some of the equations can be derived from linear combinations of others. Gaussian elimination allows us to reduce to a smallest possible set of *independent* equations with same solution space

If $m = n$ after reduction, the system either has no solution (in which case, no feasible solution to LP), or a unique solution, which may or may not be feasible (i.e., may or may not satisfy $\mathbf{x} \geq 0$). In either case, the methods of linear algebra solve the LP

So from now on, we assume that $m < n$, meaning we have more variables than constraints. This is what we most often encounter in linear programming, since $\leq$ or $\geq$ constraints in the original problem introduce new variables, but no new constraints, and the same for unrestricted variables.

Basic feasible solutions

Take any $n - m$ of the n variables and set them to zero. These variables are called the *non-basic variables*.

The matrix equation $A\mathbf{x} = \mathbf{b}$ reduces to a system of m equations in m unknown variables. These variables are called the *basic variables*.

This m by m system may have a unique solution in which all the components are non-negative. If it does, the feasible solution to the LP with the non-basic variables set to zero, and the value of the basic variables determined by the unique solution to the m by m system, is called a *basic feasible solution*.

Just as the optimal solution to a 2-variable LP occurs at a corner point of the feasible space, **the optimal solution to an LP in standard form always occurs at a basic feasible solution**.

Solving an LP in finite time

- Convert to standard form, m constraints, n variables
- Use linear algebra to make sure there is no redundancy among equations (so $m \leq n$)
- If $m = n$, solve $A\mathbf{x} = \mathbf{b}$ to perhaps get unique feasible solution
- If $m < n$:
  - For each subset of $n - m$ variables, set these variables to zero, and see if there is a unique, non-negative solution to the resulting m by m system. If there is, record this as a basic feasible solution
  - Evaluate the objective at all basic feasible solutions
  - The optimum is the best value among basic feasible solutions

Finite . . . but not necessarily quick!

How many basic feasible solutions are there? At most

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

(this is the number of ways of choosing the m potentially basic variables from the set of n variables).

How long might the procedure just described take?

- $n = 10$, $m = 5$: 252 steps
- $n = 20$, $m = 10$: 184,756 steps
- $n = 30$, $m = 15$: 155,117,520 steps
- $n = 40$, $m = 20$: 137,846,528,820 steps
- $n = 50$, $m = 25$: 126,410,606,437,752 steps

Things quickly spiral out of control!

Why are basic feasible solutions so special?

Here's a very high-level idea of what's going on:

- If we set fewer than $n - m$ components of $\mathbf{x}$ to zero, get to solve m equations in more than m variables to determine the rest of $\mathbf{x}$. There are infinitely many solutions, and around any one solution there is a "ball" of other solutions. It's always possible to find a direction to move in inside this ball that improves the objective value. Conclusion: no solution that is obtained by setting fewer than $n - m$ components to zero can be optimal
- If we set $n - m$ components of $\mathbf{x}$ to zero, and the resulting m by m system has a solution, but not a unique one, then it has infinitely many, and we can do the same thing as in the first point to conclude that a solution so obtained isn't optimal
- Conclusion: The only solutions that might potentially be optimal are the basic feasible solutions