Sensitivity analysis — changing the objective

Consider the following problem:

Maximize

$$3x - 5y + 4w$$

subject to the constraints

$$\begin{aligned} x + y + w &\leq 6 \\ x + 2y + 3w &\leq 8 \\ x + 3y + 9w &\leq 12 \end{aligned}$$

with $x, y, w \geq 0$.

We solve using the simplex algorithm.

Initial tableau:

Basic	z	x	y	w	s_1	s_2	s_3	Soln.
Max	1	-3	5	-4	0	0	0	0
s_1	0	1	1	1	1	0	0	6
s_2	0	1	2	3	0	1	0	8
x_2	0	1	3	9	0	0	1	12

Optimal tableau:

Basic	z	x	y	w	s_1	s_2	s_3	Soln.
Max	1	0	8.25	0	2.875	0	.125	18.75
x	0	1	.75	0	1.125	0	$-.125$	5.25
s_2	0	0	.5	0	$-.75$	1	$-.25$	.5
w	0	0	.25	1	$-.125$	0	.125	.75

The optimum is 18.75, reached at the point $(5.25, 0, .75)$ (in xyw space).

How do things change if we change the objective coefficients by a small amount? Well, let's consider the following modified problem.

Maximize

$$(3 + d_1)x + (-5 + d_2)y + (4 + d_3)w$$

subject to the same constraints. Here d_1, d_2 and d_3 may be positive or negative, and represent the amount by which the objective coefficients have changed.

If this new problem has its optimum at the same point as the original problem, then we solve it by starting with a slightly modified initial tableau (modified to reflect the fact that the objective coefficients have changed), and then repeating the steps that the simplex algorithm used to go from the initial tableau above to the final (optimal) tableau. That leads to the following two new tableau:

New initial tableau:

Basic	z	x	y	w	s_1	s_2	s_3	Soln.
Max	1	$-3 - d_1$	$5 - d_2$	$-4 - d_3$	0	0	0	0
s_1	0	1	1	1	1	0	0	6
s_2	0	1	2	3	0	1	0	8
x_2	0	1	3	9	0	0	1	12

New final tableau:

Basic	z	x	y	w	s_1	s_2	s_3	Soln.
Max	1	0	$8.25 + .75d_1 - d_2 + .25d_3$	0	$2.875 + 1.125d_1 - .125d_3$	0	$.125 - .125d_1 + .125d_3$	$18.75 + 5.25d_1 + .75d_3$
x	0	1	$.75$	0	1.125	0	$-.125$	5.25
s_2	0	0	$.5$	0	$-.75$	1	$-.25$	$.5$
w	0	0	$.25$	1	$-.125$	0	$.125$	$.75$

Things to observe about this new tableau:

1. Only the objective row has changed
2. Variables which had coefficient 0 in the old optimal tableau, still have coefficient 0 in the new final tableau

3. Is this new final tableau optimal? To check, we just need to see that all coefficients in the objective row are positive. That is, we are at an optimal tableau for the new problem if the following optimality conditions hold simultaneously:

$$\begin{aligned} 8.25 + .75d_1 - d_2 + .25d_3 &\geq 0 \\ 2.875 + 1.125d_1 - .125d_3 &\geq 0 \\ .125 - .125d_1 + .125d_3 &\geq 0. \end{aligned}$$

In this case, we conclude that that the changes to our objective function have kept us at the same optimal point $(5.25, 0, .75)$, and the new optimal objective value is $18.75 + 5.25d_1 + .75d_3$.

From the above collection of simultaneous inequalities, we can easily figure out how much the coefficient of x can change by in the objective, while leaving the optimum unchanged, if all the other coefficients remain unchanged. In this case we have $d_2 = d_3 = 0$ and so

$$\begin{aligned} 8.25 + .75d_1 &\geq 0 \\ 2.875 + 1.125d_1 &\geq 0 \\ .125 - .125d_1 &\geq 0. \end{aligned}$$

In other words,

$$-2.55 \leq d_1 \leq 1.$$

As long as d_1 is in this range, that is, as long as the coefficient of x in the objective stays between $.44$ and 4, the optimum will remain unchanged if the other coefficients are fixed. We can similarly get the feasible ranges for d_2 and d_3:

$$-\infty \leq d_2 \leq 8.25$$

(so the coefficient of y can vary between $-\infty$ and 3.25) and

$$-1 \leq d_3 \leq 23$$

(so the coefficient of y can vary between 3 and 27.)

Everything in this discussion remains valid for minimization problems, except that the signs in the optimality conditions all change to $\leq$.

4. Here is a relatively easy trick for computing the new values along the objective row from the old optimal tableau: add a new row along the top of the old optimal tableau, and in this row, in each column corresponding to a variable in the problem (except z), put the amount by which the coefficient of that variable has been changed in the modified objective function (easier to do than to say: above x put d_1, above y put d_2, above w put d_3, and above s_1, s_2 and s_3 put 0). Also, add a new column on the left of the old optimal tableau, and in this column, put 1 to the left of z, and to the left of each basic variable put the amount by which the coefficient of that variable has been changed in the modified objective function (so, e.g., next to x put d_1.) Now, to compute the new coefficient in the final tableau objective row of a certain variable, take the dot product of that variable's column with the new column, and subtract the entry above that variable in the new row; and to compute the new solution entry in the objective row, take the dot product of the solution column with the new column. Again, this is easier to do than to say: for example, from the picture below

			d_1	d_2	d_3	0	0	0	
	Basic	z	x	y	w	s_1	s_2	s_3	Soln.
1	Max	1	0	8.25	0	2.875	0	.125	18.75
d_1	x	0	1	.75	0	1.125	0	$-.125$	5.25
0	s_2	0	0	.5	0	$-.75$	1	$-.25$	.5
d_3	w	0	0	.25	1	$-.125$	0	.125	.75

we see that the new coefficient of x in the objective should be:

$$(1, d_1, 0, d_3) \cdot (0, 1, 0, 0) - d_1 = 0,$$

the new coefficient of y should be

$$(1, d_1, 0, d_3) \cdot (8.25, .75, .5, .25) - d_2 = 8.25 + .75d_1 - d_2 + .25d_3,$$

the new coefficient of w should be $(1, d_1, 0, d_3) \cdot (0, 0, 0, 1) - d_3 = 0$,

the new coefficient of s_1 should be $(1, d_1, 0, d_3) \cdot (2.875, 1.125, -.75, -.125) - 0 = 2.875 + 1.125d_1 - .125d_3$,

the new coefficient of s_2 should be $(1, d_1, 0, d_3) \cdot (0, 0, 1, 0) - 0 = 0$

the new coefficient of s_3 should be $(1, d_1, 0, d_3) \cdot (.125, -.125, -.25, .125) - 0 = .125 - .125d_1 + .125d_3$

and the new solution entry should be $(1, d_1, 0, d_3) \cdot (18.75, 5.25, .5, .75) = 18.75 + 5.25d_1 + .75d_3$.

Here's a worked example: Suppose, subject to the same constraints, the objective we have to maximize is

$$4x - y + 30z.$$

In this case, $d_1 = 1$, $d_2 = 4$ and $d_3 = 26$. We check that the optimality conditions *are* satisfied:

$$\begin{aligned} 8.25 + .75(1) - (4) + .25(26) = 11.5 &\geq 0 \\ 2.875 + 1.125(1) - .125(26) = .75 &\geq 0 \\ .125 - .125(1) + .125(26) = 3.25 &\geq 0. \end{aligned}$$

We conclude that the new linear programming problem has its optimum at the same point at the old one: $x = 5.25, y = 0, w = .75$, and that the new optimum objective value is $18.75 + 5.25(1) + .75(26) = 43.5$ (as result we could independently verify by TORA).

Here's a point to take note of in this example: we are increasing the coefficient of w by $d_3 = 26$, which is outside the feasible range for d_3 for the optimum to stay at the same point. If it was just the w coefficient we changed, then, the optimum would move to a new point. The fact that we are also changing the coefficients of x and y means that we can no longer rely on the individual feasible range for d_3.