Introduction to sensitivity analysis

ABC co. manufactures gizmos and widgets. Gizmos require 2 hours of work by a skilled assembler, and 1.5 hours in a polishing machine, and generate a profit of $120. Widgets require 1.5 hours of assembly and 3 hours of polishing, and generate $150 dollars profit. Per week, 140 assembler hours are currently available, and 180 hours of polishing machine time.

ABC co. uses linear programming to discover that the optimum (with respect to profit) product mix is 40 gizmos and 40 widgets per week, for a profit of $10,800.

The linear program that was solved to obtain this solution was:

Maximize $120x_1 + 150x_2$ (profit) subject to

$2x_1 + 1.5x_2 \leq 140$ (assembler constraint)

$1.5x_1 + 3x_2 \leq 180$ (polishing machine constraint)

$x_1, x_2 \geq 0$ (positive number of gizmos, widgets produced)

The story doesn't end there

Any number of scenarios might realistically play out now:

1. ABC co. discovers that it actually has 160 assembler hours available per week. How does that change the weekly profit?
2. An agency offers the services of an assembler for a 40 hour week, at a cost of $1200, and ABC co. has to decide whether to accept.
3. ABC co. is offered a polishing machine that can be used for 60 hours per week. How much should they be willing to pay?
4. Head office wants to know which is the more important priority: hiring assemblers or purchasing polishing machines?
5. A change in the cost of raw materials means that profit for gizmos and widgets changes to $140 and $160 respectively. Does this change the optimal production mix?

These questions,

and more,

will be answered using

Sensitivity analysis

Dual price of assembly hours

Optimum is currently 10,800, achieved at $(40, 40)$: intersection of $2x_1 + 1.5x_2 = 140$ and $1.5x_1 + 3x_2 = 180$.

What if 140 is changed to $140 + \Delta$?

New optimum is $10,800 + 36\Delta$, achieved at $(40 + .8\Delta, 40 - .4\Delta)$: intersection of $2x_1 + 1.5x_2 = 140 + \Delta$ and $1.5x_1 + 3x_2 = 180$.

So: each unit change in assembly hours leads to a change of \$36 in profit at optimum

This is valid as long as $\Delta \leq 100$ (after which the assembly constraint becomes redundant) and $\Delta \geq -50$ (after which the polishing constraint becomes redundant).

\$36 is called the *unit worth* of an assembly hour, or the *dual* or *shadow price*. The range $(90, 240)$ $(= (140 - 50, 140 + 100))$ is called the *feasible range* for this dual price.

Using dual prices

We may similar compute that the dual price of an hour of polishing machine time is \$32, in the feasible range $(105, 280)$
(meaning: each extra hour of machine time generates \$32 profit, each lost hour costs \$32, as long as number of hours stays between 105 and 280).

We can now answer the first four questions asked earlier:

1. ABC co. discovers that it actually has 160 assembler hours available per week. How does that change the weekly profit?

   Ans: Adding 20 hours of assembly time keeps us within the feasible range for the dual price \$36 of assembly hours. So: profit is increased by 20 times 36 = \$720.

2. An agency offers the services of an assembler for a 40 hour week, at a cost of \$1200, and ABC co. has to decide whether to accept.

   Ans: Adding 40 hours of assembly time keeps us within the feasible range for the dual price \$36 of assembly hours. So: profit is increased by 40 times 36 = \$1440. It seems worthwhile to pay \$1200 in exchange for an profit increase of \$1440.

3. ABC co. is offered a polishing machine that can be used for 60 hours per week. How much should they be willing to pay?

   Ans: Up to an increase of 100 hours, each extra hour of polishing time is worth \$32. So an extra \$60 should be worth \$1,920. ABC co. should be willing to pay anything less than this.

4. Head office wants to know which is the more important priority: hiring assemblers or purchasing polishing machines?

   Ans: The dual price of assembly hours is greater than that of machine hours, so increasing assembly hours seems like a good priority.

Changes to the objective

Slope of assembly constraint is $-\frac{4}{3}$
Slope of polishing constraint is $-\frac{1}{2}$
Slope of objective is $-\frac{4}{5}$
If objective is $c_1x_1 + c_2x_2$, slope is $-\frac{c_1}{c_2}$

As long as slope of objective remains between $-\frac{4}{3}$ and $-\frac{1}{2}$, i.e. as long as

$$\frac{1}{2} \leq \frac{c_1}{c_2} \leq \frac{4}{3},$$

optimum remains at $(40, 40)$

We may now answer the last question posed earlier:

If profit for gizmos and widgets changes to \$140 and \$160 respectively.
Does this change the optimal production mix?

Ans: Since $\frac{140}{160} = \frac{7}{8}$ lies between $\frac{1}{2}$ and $\frac{4}{3}$, the optimum production mix remains unchanged

Optimality ranges for c_1 and c_2

Assuming c_2 remains fixed at \$150, in what range can c_1 move without changing the optimum production mix?

To not change the optimum, we must have

$$\frac{1}{2} \leq \frac{c_1}{150} \leq \frac{4}{3}$$

that is

$$75 \leq c_1 \leq 200$$

This is referred to as the *optimality range* for c_1; similarly, the optimality range for c_2 is

$$90 \leq c_2 \leq 240$$