Simplex method — summary

Problem: optimize a linear objective, subject to linear constraints

1. **Step 1**: Convert to standard form:
   - variables on right-hand side, positive constant on left
   - slack variables for $\leq$ constraints
   - surplus variables for $\geq$ constraints
   - $x = x^- - x^+$ with $x^-, x^+ \geq 0$ if x unrestricted
   - in standard form, all variables ≥ 0, all constraints equalities
2. **Step 2**: Add artificial variables:
   - one for each constraint without a slack variable
3. **Step 3**: Create an objective constraint:
   - add new variable z, and add new constraint $z-$ objective $= 0$
4. **Step 4**: Form the initial tableau:
   - first column to identify basic variables
   - last column for constants on right-hand sides of constraints
   - in between, one column for each variable (beginning with z)
   - first row for labels
   - remaining rows for constraints (beginning with objective — but see Step 5 below)
5. **Step 5**: Identify the initial objective function ($z = ?$)
   - if all constraints were $\leq$, so slack variable in each constraint, use objective function from original problem
   - if there are artificial variables, and M-method is being used, objective function is original objective plus
     - large positive multiple of each artificial variable (if minimization problem)
     - large negative multiple of each artificial variable (if maximization problem)
   - if there are artificial variables, and two-phase method is being used, objective function is sum of artificial variables, and this should be minimized (whether or not original problem was minimization)

6. **Step 6**: Identify initial basic variables:
   - slack variables together with artificial variables
   - looking at constraint rows only in columns of these initial basic variables, should see permutation of columns of identity matrix
   - label each constraint row by the basic variable occurring once in that row
7. **Step 7**: Modify the z-row
   - if entry in z-row in column of basic variable is not zero, add appropriate multiple of the row in which that basic variable appears, so that entry becomes 0
   - at end of process, objective is expressed entirely in terms of non-basic variables
   - if initial basic variables consist of all slack variables, this step not necessary
8. **Step 8**: Identify an entering basic variable and pivot column:
   - maximization problem — most negative coefficient in z-row
   - minimization problem — most positive coefficient in z-row
   - break ties by choosing left-most column
   - column of entering variable is pivot column
   - if no entering variable, STOP — optimum reached
     - current basic feasible solution is optimal
     - optimal objective value is last entry (solution column) of objective row
9. **Step 9**: Identify a departing basic variable and pivot row:
   - for each non-basic variable, take ratio of entry in solution column and entry in pivot column
   - non-basic variable with smallest non-negative ratio is departing variable, and corresponding row is pivot row
   - break ties by choosing top-most column
10. **Step 10**: Pivot on pivot entry:
    - pivot entry is intersection of pivot row and pivot column
    - scale pivot row so pivot element is one
    - add multiples of pivot row to other rows (including objective row) so rest of pivot column is zero
    - change label of pivot row to that of entering variable

11. **Step 11**: Iterate:

    - repeat steps 8 through 10 until optimal is reached
    - if using M-method or all-slack starting solution, problem is completely done; if using two-phase method, go onto step 12

12. **Step 12**: Phase 2 of two-phase method:

    - as long as phase 1 of two-phase method returns minimum of zero, continue to phase 2
    - create a new initial tableau
      - objective row given by original objective of problem
      - constraint rows given by constraint rows of final tableau of phase 1, with artificial columns removed
      - initial basic variables given by basic variables at end of phase 1
    - go back to step 7

Four quirky situations

1. **Degeneracy**:

    - Geometric idea
        - can happen when more constraints meet at a point then would be normal (three or more two variable constraints, four or more three variable constraints, etc.)
        - typically one of these constraints is redundant (its removal doesn't change the feasible space)
        - at the degenerate point, one (or more) of the basic variables is zero, so in fact the same point corresponds to numerous basic feasible solutions
    - Simplex manifestation
        - occurs whenever there is a tie for departing variable
        - at next iteration, entering variable will be constrained to enter at value zero
        - simplex algorithm will move to a new basic feasible solution, but it's geometrically the same point, and the objective doesn't change
    - Implications
        - typically, slows down simplex algorithm
        - in worst case, can lead to cycling — algorithm loops, staying at the same (suboptimal) point forever
        - this is so wildly unlikely (and difficult to deal with) that no commercial implementation of simplex algorithm bothers to deal with it

2. **Alternative optima**:
   - Geometric idea
     - happens when one of the constraints which is satisfied tightly at the optimum is parallel to the objective
     - there is more than one optimal basic feasible solution, and infinitely many optimum solutions that are not basic
   - Simplex manifestation
     - when optimality is reached, one (or more) of the non-basic variables has coefficient zero in objective
     - each one can enter into the set of basic variables, without changing the objective value
   - Implications
     - gives a variety of choices for optimum, some of which may be more desirable than others
     - doesn't affect the running of the algorithm
3. **Unbounded solution**:
   - Geometric idea
     - happens when the constraints do not trap a finite region in space, but allow at least one variable to go to infinity inside feasible region
     - the objective value can be made as large (or small, if a minimization problem) as one wishes
   - Simplex manifestation
     - when ratio test is being used to determine constraints on entering variable, all ratios are either negative or infinity
     - the current entering variable is the one that can be made as large as desired
   - Implications
     - suggests that the original problem may have been poorly posed, or fed into solver incorrectly
     - simplex algorithm should be coded to stop when finding an unbounded variable

4. **Unfeasible problem**:
    - Geometric idea
        - happens when the constraints are inconsistent
        - there is no feasible point that satisfies all the constraints
        - cannot occur when all constraints are $\leq$, because all-zero solution (all-slack solution) is feasible in this case
    - Simplex manifestation
        - occurs only when M-method or two-phase method are being used
            * M-method: no matter how large M is, one of the artificial variables is always basic in optimum solution
            * two-phase method: phase 1 ends by discovering that minimum of sum of artificial variables is positive
    - Implications
        - suggests that the original problem may have been poorly posed, or fed into solver incorrectly