Introduction to Operations Research (Math 30210) Fall 2014

Homework assignments and quizzes

Instructor: David Galvin

Contents

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1 General information

In general homework will be assigned on a Wednesday, and will be due in class the following Wednesday (an exception to this will be the first written assignment, which won't appear until Friday, August 29). As they are assigned, homework problems will be added to this document, and will be posted on the website. At the same time I will make an email announcement.

I will distribute a printout of the homework problems in class each week, with plenty of blank space. The solutions that you submit should be presented **on this printout** (using the flip side of a page if necessary). You may also print out the relevant pages from the website; if you do so, you must **staple** your pages together. The grader reserves the right to

- not grade any homework that is not presented on the printout, and
- deduct points for homework that is not stapled.

The weekly homework is an important part of the course; it gives you a chance to think more deeply about the material, and to go from seeing (in lectures) to doing. It's also your opportunity to show me that you are engaging with the course topics.

Homework is an essential part of your learning in this course, so please take it very seriously. It is extremely important that you keep up with the homework, as if you do not, you may quickly fall behind in class and find yourself at a disadvantage during exams.

You should treat the homework as a learning opportunity, rather than something you need to get out of the way. Reread, revise, and polish your solutions until they are correct, concise, efficient, and elegant. This will really deepen your understanding of the material. I encourage you to talk with your colleagues about homework problems, but your final write-up must be your own work.

Homework solutions should be complete (and in particular presented in complete sentences), with all significant steps justified. The grader reserves the right to

- not grade any homework that is disorganized and incoherent.

2 Homework 1 (due Sept. 3) Name:_______________

The purpose of this homework is to illustrate the variety of situations that can be modelled with a linear objective functions and linear constraint. [It's implicit in all the questions that "formulate mathematically" means "formulate as the problem of maximizing/minimizing a **linear** objective function, subject to a collection of **linear** constraints on the variables"]

Reading: Sections 1.1 1.2 and 1.4 (first half of page), and 2.1, 2.2, 2.3, 2.4 and 2.6.

1. Problem set 2.2, question 7 (page 18)

    - part (a) [note that you are required both to set up and solve the problem]

- part (b)

- part (c)

2. Problem set 2.2, question 14 (page 20) [Be careful! If you find that using 0 of each product is feasible, you haven't set the problem up correctly! Also, note that this question only requires you to set up the problem]

3. Problem set 2.3, question 10 (page 28) [note that you are required both to set up and solve the problem]

4. Problem set 2.3, question 15 (page 30) [note that you are required only to set up the problem]

5. Problem set 2.4, question 4 (page 37) [note that you are required only to set up the problem]

6. Problem set 2.4, question 7 (page 38) [note that you are required only to set up the problem]

7. **Extra credit problem!** Set up (just set up) the two problems concerning queens on a chessboard (from the handout on August 27, also on the website) as problems of optimizing some linear function of some variables, subject to some linear constraints (and perhaps some constraints to say that some variables are integers). So that things don't get ridiculously out-of-hand, just do the problems for a 3-by-3 chessboard.

3 Homework 2 (due Sept. 10) Name:________________

The purpose of this homework is to examine the *standard form* of a linear programming problem, and the pivot operation that allows one to find a basic feasible solution to a problem.

Reading: Sections 3.1 and 3.2.

1. Problem set 3.1, question 2 (page 60)

2. Problem set 3.1, question 3 (page 60)

   - part (a)

   - part (e)

   - part (g)

3. Problem set 3.1, question 4 (page 61)

   - part (a)

   - part (b) [for this part, illustrate the set of feasible points on a graph]

4. Problem set 3.1, question 7 (page 62) [here's what this question is asking: show that for any feasible point for Problem A, there's a corresponding feasible point for Problem B with the same objective value, and vice-versa. This problem shows that if you have a bunch unrestricted variables, you can put them into standard form simultaneously with the addition of a single new variable.]

5. Problem set 3.2, question 1 (page 70)

- part (a)

- part (b)

6. Problem set 3.2, question 2 (page 71)

7. Problem set 3.2, question 3 (page 71)

8. Problem set 3.2, question 6 (page 72)

- part (a)

- part (b)

- part (c)

- part (d)

9. Problem set 3.2, question 9 (page 72)

4 Homework 3 (due Sept. 17) Name:________________

The purpose of this homework is to explore the simplex algorithm for solving a linear programming problem in standard form.

Reading: Sections 3.3 and 3.4

1. Problem set 3.3, question 1 (page 76)

    - part (a)

    - part (b)

    - part (c)

- part (d)

- part (e)

- part (f)

2. Problem set 3.3, question 2 (page 76)

   - part (a)

   - part (b)

- part (c)

3. Problem set 3.3, question 3 (page 77)

4. Problem set 3.3, question 4 (page 77)

    - part (a)

    - part (b)

5. Problem set 3.4, question 2 (page 83)

    - part (a)

- part (f)

6. Problem set 3.4, question 6 (page 84)

5 Quiz 1 (Sept. 12) Name:_______________

1. Formulate the following problem as a linear programming (LP) problem. Say what each of your variables represents, state the objective function and whether it is to be minimized or maximized, and state **all** constraints that must be imposed on the variables.

   A factory has two machines to make stuff. Machine one uses 80lbs of raw material per day of operation, requires 16 hours of labour per day, and produces 37lbs of stuff per day. Machine two uses 50lbs of raw material per day of operation, requires 35 hours of labour per day, and produces 43lbs of stuff per day. It is required that exactly 200lbs of stuff is produced per week (which can consists of up to seven full days of each machine running). Up to 300lbs of raw material can be purchased from supplier A at $4 per lb, and an unlimited amount of raw material can be purchased from supplier B at $5 per lb. 150 hours of labour is available at $8 per hour, and an additional 30 hours of overtime labour is available at $12 per hour. Only labour and raw materials used are paid for. It is allowable to purchase fractional lbs of raw material, use fractional hours of labour, and run each machine for a fractional number of days. What is the minimum cost required? [DON'T SOLVE!!! JUST SET UP!]

2. Put the following LP problem in standard form: Maximize $a + b + c$ subject to

$$3a - 2b + 4c \leq 2$$

as well as $a \geq 0$, $b \geq 1$ and $c \leq 1$.

6 Homework 4 (due Sept. 26) Name:________________

Note that the homework is due on **FRIDAY**! The purpose of this homework is to explore the tableau presentation of the simplex algorithm, and the method of artificial variables for generating an initial basic feasible solution. A separate part of this homework also explores Solver, the Excel LP tool.

General note on presenting solutions: The complete solution to an LP consists EITHER of the statement that the problem is unbounded, OR the statement that it has no feasible solution, OR the statement that it has a particular optimum value, achieved at a particular point. When I say "write the solution to the problem in the space provided", this is what I am asking for. If you use LP Assistant (which you will have to for some of these problems), you should take a screenshot of the LP Assistant tableaux and include it with the homework you turn in. PLEASE be careful to label your screenshot printouts in such a way that it is easy to match the printouts with the problems! Something like "Q 11 part (p)" written prominently on the screenshot printout will do.

Reading: Sections 3.5 and 3.6, and the handout on Solver.

1. Problem set 3.5, question 1 (page 89). For each part, reproduce the tableau and either circle all the entries on which is it valid to pivot, or say that the problem is completely resolved (with specifics: either the solution is unbounded, or a particular optimum is achieved by a particular solution). DON'T use LP Assistant!

    - part (a)

- part (d)

- part (e)

- part (f)

2. Problem set 3.5, question 2 (page 90). Feel free to ignore the instructions, and use LP Assistant. Print out a screenshot of the LP Assistant solution, mark clearly which question it belongs to, and in the space below write the solution to the problem (optimum, point achieving optimum). Be careful getting the problems into canonical form!

   - part (a)

   - part (d)

   - part (f)

3. Problem set 3.6, question 1 (page 99), part (a). DON'T use LP Assistant!!!!

4. Problem set 3.6, question 2 (page 99). For these three parts, please DO use LP Assistant. Print out a screenshot of the LP Assistant solution, and in the space below write the solution.

   - part (a)

   - part (d)

   - part (f)

5. Problem set 3.6, question 3 (page 100) (Setup the problem in the space below. Then solve it using LP Assistant).

6. Problem set 3.6, question 5 (page 101)

7 Homework 5 (due Oct. 3) Name:_______________

Note that the homework is due on **FRIDAY**! The purpose of this homework is to explore the how artificial variables expose redundancy in a system of equations, to practice using the full simplex method, and to explore the concept of duality.

Reading: Sections 3.7, 4.1, 4.2 and 4.3.

1. Problem set 3.7, question 2 (page 105).

   - part (a)

   - part (b) (For this part, if you use LP Assistant then print out your tableau and explain what part of the tableau exposes the redundancy).

2. Problem set 3.7, question 3 (page 105). For both of these, if you use LP Assistant then print out your tableau. In the space below present the required answers.

   - part (d)

   - part (g)

3. Problem set 4.2, question 1 (page 130).

   - part (a)

   - part (c)

   - part (f)

4. Problem set 4.2, question 2, part (d) (page 131).

5. Problem set 4.2, question 3 (page 131)

- part (a)

- part (b)

- part (c) (If you use LP Assistant then print out your tableau. In the space below present the required answers.)

- part (d)

6. Problem set 4.3, question 3 (page 136)

- part (a)

- part (b)

- part (c) (If you use LP Assistant then print out your tableau. In the space below present the required answers.)

- part (d)

7. Problem set 4.3, question 5 (page 138)

- part (a)

- part (b)

- part (c)

8 Quiz 2 (Oct. 1) Name:________________

1. The tableau below reveals that the linear programming currently being solved is not bounded from below.

Basis	x_1	x_2	x_3	x_4	x_5	x_6	rhs
x_3	2	0	1	0	0	2	12
x_4	-4	-3	0	1	0	-1	6
x_5	3	-5	0	0	1	3	20
Obj	5	-2	0	0	0	-2	40

- Part (a): Suppose that x_2 enters the set of basic variables, with some value X, while all the other non-basic variables (x_1, x_6) remain at value 0. What value (it will depend on X) should the basic variable x_4 be set to, to maintain feasibility?

- Part (b): Again, Suppose that x_2 enters the set of basic variables, with some value X, while all the other non-basic variables remain at value 0. What will be the new value of the objective function? (It will depend on X.)

- Part (c): Give an example of a feasible point for this problem that has objective value -240.

2. Consider the following LP problem: Minimize $a + b + c$ subject to

$$\begin{aligned} 3a - 2b - c &\leq 2 \\ -a + 4b + c &\geq 4 \\ 2a + 3b - 2c &= 4 \end{aligned}$$

as well as $a, b, c \geq 0$.

- Set up the initial simplex tableau for this problem, including all necessary slack variables and artificial variables. If using artificial variables, be sure to correctly present the artificial objective function.

- Say which of the following three things will happen when you begin running the simplex algorithm on your tableau:
  - **A**: The algorithm immediately terminates with "optimum reached" (in this case, say what the optimum is, and what point it is reached at)
  - **B**: The algorithm terminates with "the problem is not bounded from below" (in this case, circle the entries of the tableau that allow you to conclude this)
  - **C**: A pivoting occurs (in this case, circle all entries in the tableau on which it would be legitimate to pivot)

9 Homework 6 (due Oct. 13) Name:_______________

Note that the homework is due on **MONDAY**! The purpose of this homework is to explore the duality theorem.

Reading: Section 4.4.

1. Problem set 4.4, question 1 (page 149).

2. Problem set 4.4, question 2 (page 149).

3. Problem set 4.4, question 4 (page 149). [Your solution to this problem should **not** involve any attempt to solve either the primal or the dual via simplex method!]

4. Problem set 4.4, question 6 (page 150).

- part (a)

- part (b)

- part (c)

- part (d)

- part (e)

- part (f)

5. Problem set 4.4, question 7 (page 150). ["Solve" here means, as usual, give the optimum value of the objective of the given problem, and say what point that optimum is achieved at; or day that the problem is unbounded or has no feasible points. Note the instruction to solve the given problem by using the simplex algorithm on the *dual* problem, so no credit for solving the given problem directly. If you use LP Assistant, just print out a screenshot of your final tableau (but be sure to write the full solution to the question below).]

6. Problem set 4.4, question 10 (page 151)

   - part (a)

- part (b)

- part (c)

10 Midsemester exam (Oct. 15) Name:_______________

This exam is for Math 30210, Fall 2014.
This exam contains 5 problems on 6 pages (including the front cover).
This is an open-book exam. You may use the textbook, old homeworks, notes, etc.
You may use a calculator.
For full credit **show all your work** on the paper provided.
The honor code is in effect for this exam.

Scores

Question	Score	Out of
1		10
2		10
3		10
4		10
5		10
Total		50

MAY THE ODDS BE EVER IN YOUR FAVOR!

1. Set up the following problem as a linear programming problem. Be sure to

   - say what each variable represents
   - say what the objective function is, and whether it is to be maximized or minimized
   - list all constraints on the variables

   Don't solve!

   A factory can produce units of each of items A and B. A unit of each item has an associated profit in dollars, and uses a number of labor hours and a number of pounds of raw material. The relevant data is summarized in the table below:

Data	A	B	Availability
Labor	3	4	100
Materials	2	1	80
Profit	20	30	

   Any leftover materials have to be stored at a cost of \$4 per lb. Contractual obligations require that the number of units of B made is at least twice the number of units of A made. Find a production scheme that maximizes revenue (profit minus storage).

2. Put the following problem in standard form:

   Maximize $-x_1 - x_2 + 2x_3 + x_4$ subject to

$$\begin{aligned} 7x_1 + 16x_2 &\leq 4x_3 + x_4 \\ x_2 + 12x_3 &= 6x_1 \\ 9x_4 &\geq x_1 + 3x_3 \end{aligned}$$

   and $x_1, x_2 \geq 0$, x_3 unrestricted, and $0 \leq x_4 \leq 10$.

3. (a) Set up the initial tableau for the following problem (this may require the introduction of slack and/or artificial variables; if using artificial variables be sure to correctly modify the artificial objective). Clearly identify (using the left-hand column of the initial tableau) what the initial basic feasible solution is.

Maximize $3x_1 + 2x_2 + 4x_3$ subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 4 \\ 2x_1 + 3x_2 + x_3 &= 6 \\ x_1 - x_2 + 3x_3 &\geq 4 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$.

(b) If set up correctly, the simplex algorithm of the problem from the previous part should begin with a pivot. Circle all entries on the tableau on which it is legitimate to pivot.

4. Consider the following linear programming problem:

Minimize $x_1 + x_2 + x_3$ subject to

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &\geq 0 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 - x_2 - 3x_3 &\leq 4 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$.

(a) Write down the dual of this problem.

(b) I believe that $x_1 = 1$, $x_2 = 0$ and $x_3 = 0$ is a solution to the minimization problem. By considering the dual point at which the dual variables corresponding to the first and third primal constraints are both 0, and the dual variable corresponding to the second primal constraint is 1/2, verify that my belief is justified, and write down the optimum objective value. Say clearly what result you are using.

5. The tableau below was encountered while trying to solve a particular linear programming problem with three constraints and variables a, b, c, d, e and f.

Basis	a	b	c	d	e	f	rhs
c	2	0	1	0	0	2	12
e	-4	-3	0	0	1	-1	6
d	3	5	0	1	0	3	20
Obj	5	2	0	0	0	2	40

(a) Explain *carefully* why this tableau shows that the simplex method has ended, and that an optimal solution to the problem has been reached. [It is not enough just the state the optimality criterion, and observe that the above tableau satisfies the criterion; you need to explain why, for this particular example, the optimality criterion is correct].

(b) Assuming that the original problem being solved was a maximization problem, write down the basic feasible point at which the optimum for the problem is reached, and what the objective value is at that point.

11 Homework 7 (due Oct. 31) Name:________________

The purpose of this homework is to explore the use of integer constraints in linear programming problems, and in particular the use of auxiliary variables.

Reading: Sections 6.1 and 6.2.

1. Problem set 6.2, question 5 (page 224). [Just the first and third figures!]

    - Figure 6.3

- Figure 6.5

2. Problem set 6.2, question 6 (page 225). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

3. Problem set 6.2, question 9 (page 225). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

4. Problem set 6.2, question 13 (page 227). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

5. Problem set 6.2, question 15 (page 227). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it. Note also that two constraints together in the same set of braces means that BOTH of those constraints should be satisfied.]

6. Problem set 6.2, question 18 (page 227). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

7. Problem set 6.2, question 1 (page 223).

    - part (a)

    - part (b)

- part (c) [Note that two constraints together in the same set of braces means that BOTH of those constraints should be satisfied.]

12 Homework 8 (due Nov. 10) Name:______________

The purpose of this homework is to explore the cutting plane & branch/bound algorithms for solving integer programming problems.

Reading: Sections 6.3 and 6.4.

1. Problem set 6.3, question 1, part a) (page 236). [Just part a)]

2. Problem set 6.3, question 2, part c) (page 236). [Just part c)]

3. Problem set 6.3, question 4 (page 237).

4. Problem set 6.3, question 5 (page 237).

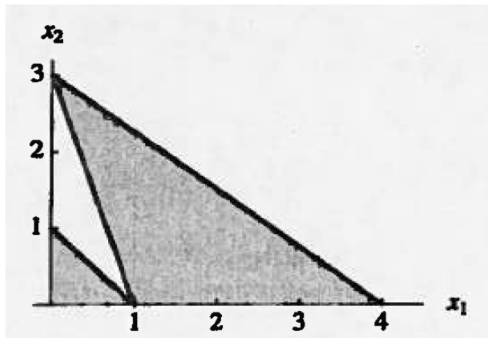
5. Problem set 6.4, question 2, part a) (page 244).

6. Problem set 6.4, question 2 part c) (page 244).

13 Quiz 3 (Nov. 10) Name:_______________

All questions relate to the shaded region of the x_1-x_2 plane shown below. It's the region of points in the x_1-x_2 plane satisfying $x_1 \geq 0$, $x_2 \geq 0$ and

- EITHER [constraint 1]
- OR both [constraint 2] AND [constraint 3].



1. Write down the actual inequalities [constraint 1], [constraint 2] and [constraint 3]. Be sure to say which is [constraint 1].

2. By introducing some auxiliary variables, describe the shaded region using a system of linear constraints. Be sure to write down *all* the necessary constraints to describe the region.

14 Homework 9 (due Nov. 21) Name:________________

The purpose of this homework is to implement & explore the transportation algorithm, and Kruskal's minimum spanning tree algorithm.

Reading: Class notes and slides on both algorithms

1. Problem set 7.2, question 1, part a) (page 277). [Just part a)]

2. Problem set 7.2, question 3, part c) (page 278). [Just part c); note that supply does not equal demand here, so a suitable modification needs to be made]. **JUST DO THE FOLLOWING**: set the problem up as a transportation problem. Start with the NW corner basic feasible solution, and run enough iterations of the algorithm that you end up with a basic feasible solution that does not use any of the links that have an infinite cost [i.e., keep going just until you have a solution that is feasible for the original problem]. Run the optimality criterion on this solution, but don't do any further looping.

3. Suppose you have a transportation problem with 3 warehouses A, B and C, having supplies s_A, s_B and s_C, and 4 outlets 1, 2, 3 and 4, having demands d_1, d_2, d_3 and d_4. Suppose further that supply exceeds demand by a certain surplus S. Explain how you would modify the cost array so that the following problem variations can be solved using the transportation algorithm:

   (a) If there are any surplus units at warehouse A, they must be stored at a cost of c per unit.

   (b) If there are any surplus units at warehouse A, they can (and will) be sold at a price of r per unit.

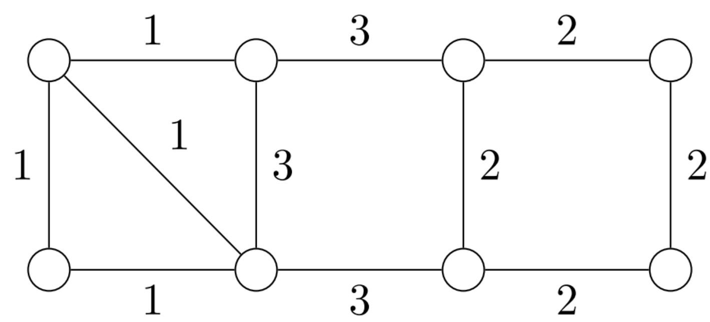
   (c) It is not permitted to have any surplus units at warehouse A.

4. A legal firm has accepted four new cases, each of which can be adequately handled by any one of the firm's four junior partners. Due to differences in experience and expertise, the junior partners would spend varying amounts of time on the cases. A senior partner has estimated the time requirements (in hours), and this information is tabulated below. Use the transportation algorithm to solve the problem of assigning cases, one per junior lawyer, in such a way as to minimize the total hours expended.

	Case 1	**Case 2**	**Case 3**	**Case 4**
Lawyer 1	14	12	13	9
Lawyer 2	8	6	8	4
Lawyer 3	12	10	9	6
Lawyer 4	11	8	11	8

JUST DO THE FOLLOWING: Set the problem up as a transportation problem. Run two iterations of the transportation algorithm; if you haven't reached optimality at this point, then stop. [It turns out that this is a problem with an awful lot of degeneracy, that runs a serious risk of exhibiting cycling].

5. The picture below shows 8 cities (represented by circles), and the cost of connecting up each pair directly (if there is no line joining two cities, then it is not possible to join the cities up directly). Use Kruskal's algorithm to find a minimum-cost scheme to connect up the 8 cities.



6. • There are 6 cities, numbered 1 through 6, and the cost of connecting city i to city j is $i + j$. Use Kruskal's algorithm to find a minimum-cost scheme to connect up the 6 cities.

- Based on your answer to the last part, give a suggestion for what connection scheme has minimum cost, when there are n cities, numbered 1 through n, and the cost of connecting city i to city j is $i + j$. Show that your suggestion yields the correct answer when $n = 3, 4$ and 5.

7. You are given a list of 100 cities, and for each pair of cities, you are told whether or not Delta airlines has a non-stop connection between those cities. You are asked to determine the answer to the following question: does Delta's route network connect up these 100 cities (when multi-city flights are allowed)? Explain how you could use Kruskal's algorithm (and the given data) to answer the question.

15 Homework 10 (due Dec. 8) Name:_______________

The purpose of this homework is to explore some aspects of two-person zero-sum games.

Reading: Sections 9.1 — 9.5

1. Problem set 9.1, question 1, part c) (page 343). [Just part c)]

2. Problem set 9.1, question 2, part a) (page 343). [Just part a)]

3. Problem set 9.3, question 1, parts a) and b) only (page 352).

   - part a)

   - part b)

4. Problem set 9.3, question 2, parts c) and e) only (page 353).

    - part c)

    - part e)

5. Problem set 9.3, question 3 (page 353)

6. Problem set 9.4, question 2 (page 359)

- part a)

- part b)

- part c)

7. Problem set 9.4, question 4 (page 360)

8. Problem set 9.4, question 6 (page 360)

9. Problem set 9.5, question 5, parts c) and d) only (page 369).

- part c)

- part d)

10. Problem set 9.5, question 16 (page 370)

11. This question relates to Problem set 9.1, question 1, part e).

    - Write down the payoff matrix for this game (Colonel Blotto is player 1).

    - Write down the linear programming problem that Colonel Blotto has to solve, to determine his optimal security mixed strategy.

    - Using any method you wish (LP Assistant, Solver, simplex by hand, magic,...), determine Colonel Blotto's optimal security mixed strategy, and the value of the game.