15 Homework 10 (due Dec. 8) Name: SOLUTIONS

The purpose of this homework is to explore some aspects of two-person zero-sum games.

Reading: Sections 9.1 — 9.5

1. Problem set 9.1, question 1, part c) (page 343). [Just part c)]

Possible strategies for P_1: select 1, 2, 3, 4

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ s_1 & s_2 & s_3 & s_4 \end{array}$$

" " " P_2: guess 1, 2, 3, 4

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ t_1 & t_2 & t_3 & t_4 \end{array}$$

If P_1 plays s_1, P_2 plays t_1, then P_2's guess is correct, P_2 gets 2 from P_1, so $a_{11} = -2$

Similar reasoning leads to payoff matrix:

	t_1	t_2	t_3	t_4
s_1	-2	0	1	1
s_2	0	-4	2	2
s_3	3	3	-6	0
s_4	4	4	0	-8

2. Problem set 9.1, question 2, part a) (page 343). [Just part a)]

P_1 has 3 ways to play his first card (K, Q, J) and 2 subsequent ways to play his second (Q, J if first was K, etc.). After that, P_1 has no choices left. So P_1 has <u>six</u> possible strategies, encodable by:

KQ ← meaning, play K first, then Q, and then (if necessary) play J
KJ
QK
QJ
JK
JQ

Similarly P_2 has 6 possible strategies; so the payoff matrix is 6×6.

3. Problem set 9.3, question 1, parts a) and b) only (page 352).

- part a)

$$\begin{array}{r c l} & & \text{row min} \\ & \begin{bmatrix} 9 & 7 & 8 & 10 \\ 6 & 5 & 12 & 8 \\ 8 & 10 & 5 & 9 \end{bmatrix} & \begin{matrix} 7 \\ 5 \\ 5 \end{matrix} \\ \text{Col Max} & \begin{matrix} 9 & 10 & 12 & 10 \end{matrix} & \end{array}$$

Max of row min; $v_1 = 7$

Min of Col Max; $v_2 = 9$

No row min is simultaneously a col max, so no saddle point [equiv: $v_1 \neq v_2$]

- part b)

$$\begin{array}{r c l} & & \text{row min} \\ & \begin{bmatrix} 2 & 6 & 1 & 2 \\ 3 & 5 & 4 & 3 \\ 1 & 0 & 2 & 4 \end{bmatrix} & \begin{matrix} 1 \\ 3 \\ 0 \end{matrix} \\ \text{Col max} & \begin{matrix} 3 & 6 & 4 & 4 \end{matrix} & \end{array}$$

v_1, max of row min

v_2, Min of Col Max

The row 2, col 1 entry (circled) is simultaneously a row min and column max, so is a saddle point. Value of game is 3, Solution is (s_2, t_1).

4. Problem set 9.3, question 2, parts c) and e) only (page 353).

- part c)

$\begin{bmatrix} x & 1 \\ 3 & x \end{bmatrix}$.

Case i) $x < 1$, row min are x, x, col max are $3, 1$, max row min < min col max, no saddle point

Case ii), $x = 1$, max row min $= 1$, min col max $= 1$, a_{12} and a_{22} both saddle points

Case iii) $1 < x < 3$, max row min $= x$, min col max $= x$, a_{22} is only saddle point

Case iv) $x = 3$, max row min $= 3$, min col max $= 3$, a_{21}, a_{22} both saddle points

Case v) $x > 3$, max row min $= 3$, min col max $= x$, no saddle point.

Summary:

) $1 < x < 3$, unique saddle point at a_{22}

) $x = 1$, saddle points at a_{11}, a_{22}

) $x = 3$, saddle points at a_{21}, a_{22}

- part e)

Using a similar case-by-case reasoning, get that entry a_{11} is a saddle point for <u>all</u> x.

5. Problem set 9.3, question 3 (page 353)

Suppose a_{ij} and a_{kl} are saddle points, with
$i \neq k$ and $j \neq l$

WLOG $i < k$ and $j < l$.

$$\begin{array}{c c} & \begin{array}{cc} j & l \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{bmatrix} \end{array}$$

a_{ij} is a row min, so
$a_{ij} \leq a_{il}$

a_{kl} is a col max, so
$a_{il} \leq a_{kl}$

a_{kl} is a row min, so
$a_{kl} \leq a_{kj}$

a_{ij} is a row Max, so
$a_{kj} \leq a_{ij}$

So $a_{ij} \leq a_{il} \leq a_{kl} \leq a_{kj} \leq a_{ij}$. Since start and end of this chain are equal, all $\leq$'s must be $=$'s, so $a_{ij} = a_{kl}$

What about two points in the same row, eg a_{ij} and a_{il}, $j < l$
If both are saddle points, both are min in their rows, but since they are in the same row they must be equal.
Similarly for two points in the same column.

6. Problem set 9.4, question 2 (page 359)

- part a)

$$\left[\tfrac{2}{3}\ \tfrac{1}{3}\ 0\right]\begin{bmatrix} -1 & 1 & 2 & 0 \\ 4 & -2 & -3 & 2 \\ 0 & 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\left[\tfrac{2}{3}\ 0\ \tfrac{1}{3}\ \tfrac{2}{3}\right]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}$$

Min of this over all of P_2's pure strategies is $\underline{0}$ [it's just equal to min of entries of row vector]

Summary:
Security level of both proposed strategies is 0

$$\left[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right] A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \left[1\ \tfrac{2}{3}\ 0\ 0\right]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$ Min over P_2's pure strategies is 0

- part b)

$$[x_1\ x_2\ x_3] A \begin{bmatrix} \tfrac{1}{4} \\ \tfrac{1}{4} \\ \tfrac{1}{4} \\ \tfrac{1}{4} \end{bmatrix} = [x_1\ x_2\ x_3]\begin{bmatrix} \tfrac{1}{2} \\ \tfrac{1}{4} \\ \tfrac{1}{2} \end{bmatrix}$$ Max over P_1's pure strategies is $\tfrac{1}{2}$

$$[x_1\ x_2\ x_3] A \begin{bmatrix} 0 \\ \tfrac{1}{2} \\ 0 \\ \tfrac{1}{2} \end{bmatrix} = [x_1\ x_2\ x_3]\begin{bmatrix} \tfrac{1}{2} \\ 0 \\ \tfrac{1}{2} \end{bmatrix}$$ Max is $\tfrac{1}{2}$

Summary: security level of both proposed strategies is $\tfrac{1}{2}$

- part c)

Since v_1 = max over all strategies of security level, can concluded that $v_1 \geq 0$

Similarly, can conclude that $v_2 \leq \tfrac{1}{2}$.

7. Problem set 9.4, question 4 (page 360)

For P_1 strategy (x_1, x_2), payoff if P_2 plays t_1 is $-x_1 + 3x_2$

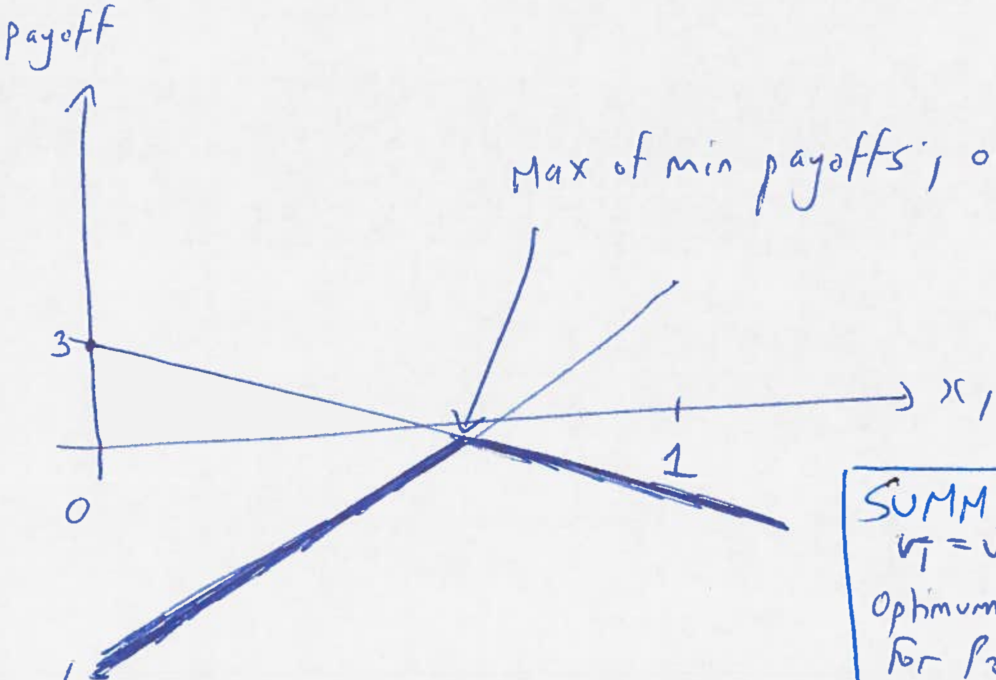
$= -x_1 + 3(1-x_1)$

$= 3 - 4x_1$

payoff if P_2 plays t_2 is $4x_1 - 6x_2$

$= 4x_1 - 6(1-x_1)$

$= 10x_1 - 6$



Max of min payoffs; occurs when

$3 - 4x_1 = 10x_1 - 6$

$x_1 = \frac{9}{14}$ $\left(x_2 = \frac{5}{14}\right)$

payoff $= \frac{3}{7}$

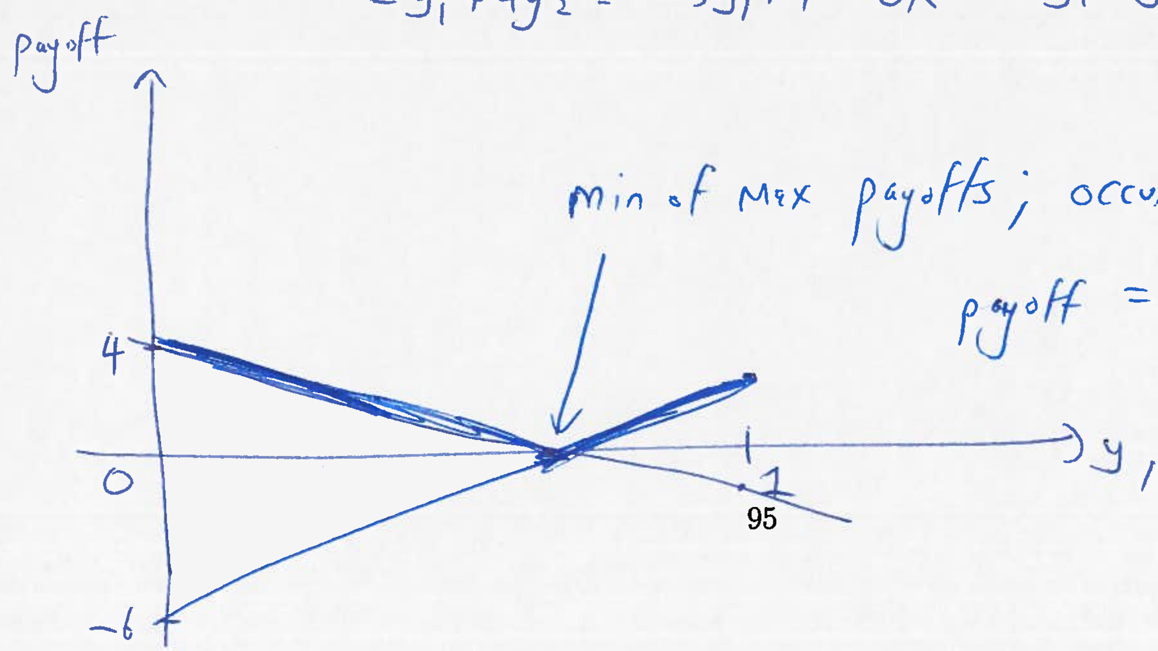
SUMMARY: $v_1 = v_2 = \frac{3}{7}$

Optimum strategy for P_1: $\left(\frac{9}{14}, \frac{5}{14}\right)$

for P_2: $\left(\frac{5}{7}, \frac{2}{7}\right)$. P_1 has advantage

For P_2: playing (y_1, y_2), payoffs are either

$-y_1 + 4y_2 = -5y_1 + 4$ OR $3y_1 - 6y_2 = 9y_1 - 6$



min of Max payoffs; occurs when $y_1 = \frac{5}{7}$ $\left(\text{and } y_2 = \frac{2}{7}\right)$,

payoff $= \frac{3}{7}$

8. Problem set 9.4, question 6 (page 360)

v_1 is defined by $\max_{X \in S} \min_{Y \in T} XAY^t$.

But Thm 9.4.1 says that the inner minimization need only be done over all pure strategies for P_2:

$$v_1 = \max_{X \in S} \left\{ \begin{array}{l} \text{min payoff to } P_1 \text{, minimized over} \\ P_2\text{'s pure strategies} \end{array} \right\}$$

OTOH, U_1 is defined as:

$$\begin{array}{l} \text{max over all} \\ \text{pure strategies} \\ \text{for } P_1 \end{array} \left\{ \begin{array}{l} \text{Min payoff to } P_1 \text{, minimized} \\ \text{over } P_2\text{'s pure strategies} \end{array} \right\}$$

v_1, U_1 are maximizing the same function, but v_1 is doing the maximizing over a larger set of inputs [all possible strategies, mixed and pure, as opposed to all pure strategies]

Hence v_1 is at least as large as U_1.

An almost identical proof shows $v_2 \le U_2$.

9. Problem set 9.5, question 5, parts c) and d) only (page 369).

- part c)

Security level of $(\frac{1}{2}, 0, \frac{1}{2})$ for P_1 is min of $-\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}$, so is $-\frac{1}{2}$; so $v_1 \geq -\frac{1}{2}$

Security level of $(\frac{1}{6}, 0, 0, \frac{5}{6})$ for P_2 is max of $-\frac{1}{2}, -\frac{2}{3}, -\frac{1}{2}$, so is $-\frac{1}{2}$; so $v_2 \leq -\frac{1}{2}$

Since $v_1 = v_2$, must have $v_1 = v_2 = -\frac{1}{2}$, and this pair of strategies does form a solution

- part d)

Security level for P_1 of $(\frac{2}{5}, \frac{3}{5}, 0)$ is $-\frac{1}{5}$, so $v_1 \geq -\frac{1}{5}$

Security level for P_2 of $(\frac{1}{9}, \frac{2}{3}, \frac{1}{9}, \frac{1}{9})$ is $\frac{4}{9}$, so $v_2 \leq \frac{4}{9}$

Here $-\frac{1}{5} \neq \frac{4}{9}$ so at least one player has not yet reached optimum security level, and the proposed pair is <u>not</u> a solution.

10. Problem set 9.5, question 16 (page 370)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Since $A^t\,(n \times m) = -A\,(m \times n)$, know $m = n$.

Know also that $a_{ii} = -a_{ii}$, so $a_{ii} = 0 \ \forall i$, and that for $i \neq j$, $a_{ij} = -a_{ji}$

So $$A = \begin{bmatrix} a_1 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & \\ \vdots & & & \\ -a_{1n} & \vdots & \cdots & 0 \end{bmatrix}$$

Security level of strategy $(x_1, \ldots, x_n)$ for P_1:

$$\min \left\{ \begin{array}{l} -a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n, \\ a_{12}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n, \\ a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - \ldots - a_{3n}x_n, \ldots, \\ a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{n-1\,n}x_{n-1} \end{array} \right\}$$ Call this s

Security level of strategy $(x_1, \ldots, x_n)$ for P_2

$$\max \left\{ \begin{array}{l} a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n, \\ -a_{12}x_1 + a_{23}x_3 + \ldots + a_{2n}x_n, \\ -a_{13}x_1 - x_{23}x_2 + a_{34}x_4 + \ldots + a_{3n}x_n, \\ -a_{1n}x_1 - a_{2n}x_2 - \ldots - a_{n-1\,n}x_{n-1} \end{array} \right\}$$ Call this t

s is $\min\{$bunch of things$\}$; t is $\max\{$negatives of those things$\}$

So (key point) $\boxed{t = -s}$

Now have $s \leq v_1 = v_2 \leq t = -s$, so $s \leq -s$

This means that $\boxed{s \leq 0}$ and so $\boxed{v_1 \leq 0}$ (since s was security of arbitrary strategy)

But also $-s \geq 0$, so $\boxed{v_2 \geq 0}$

Since $v_1 = v_2$, must have $\boxed{v_1 = v_2 = 0}$

11. This question relates to Problem set 9.1, question 1, part e).

- Write down the payoff matrix for this game (Colonel Blotto is player 1).

$$
\begin{array}{cc|ccccc}
 & & (4,0) & (3,1) & (2,2) & (1,3) & (0,4) \leftarrow P_2 \\
\hline
 & (3,0) & -3 & 0 & 4 & 3 & 2 \\
P_1 & (2,1) & 0 & -2 & -1 & 2 & 1 \\
 & (1,2) & 1 & 2 & -1 & -2 & 0 \\
 & (0,3) & 2 & 3 & 4 & 0 & -3
\end{array}
$$

- Write down the linear programming problem that Colonel Blotto has to solve, to determine his optimal security mixed strategy.

To construct LP, need positive payoffs; add 4 to each entry (eg)

LP1 : $\min \frac{1}{w} = x_1' + x_2' + x_3' + x_4'$ subject to

$$x_1' + 4x_2' + 5x_3' + 6x_4' \geq 1$$
$$4x_1' + 2x_2' + 6x_3' + 7x_4' \geq 1$$
$$8x_1' + 3x_2' + 3x_3' + 8x_4' \geq 1$$
$$7x_1' + 6x_2' + 2x_3' + 4x_4' \geq 1$$
$$6x_1' + 5x_2' + 4x_3' + x_4' \geq 1$$

all $x_i' \geq 0$

- Using any method you wish (LP Assistant, Solver, simplex by hand, magic,...), determine Colonel Blotto's optimal security mixed strategy, and the value of the game.

Solution: $\frac{1}{w} = \frac{4}{17}$, $w = \frac{17}{4}$, $x_1' = \frac{9}{289}$, $x_2' = \frac{23}{289}$, $x_3' = \frac{28}{289}$, $x_4' = \frac{8}{289}$,

So X_0 = optimal security strategy $= \left(\frac{9}{68}, \frac{23}{68}, \frac{28}{68}, \frac{8}{68}\right)$

Value of game is $\frac{17}{4} - 4 = \boxed{\frac{1}{4}}$

↑ subtract off r!