3 Homework 2 (due Sept. 10) Name: SOLUTIONS

The purpose of this homework is to examine the *standard form* of a linear programming problem, and the pivot operation that allows one to find a basic feasible solution to a problem.

Reading: Sections 3.1 and 3.2.

1. Problem set 3.1, question 2 (page 60)

As the hint observes, since $3+3 \le 6$, $3+2.3 \le 10$, $3,3 \ge 0$,

$(x_1, x_2) = (3, 3)$ satisfies inequalities of set A

But there is no $x_3 \ge 0$ s.t. $(x_1, x_2, x_3) = (3, 3, x_3)$ satisfies the equalities of set B; if there was, then x_3 would satisfy

$$\left.\begin{array}{ll} 3+3+x_3 = 6 & \text{or } x_3 = 0 \\ \text{and } 3+2.3+x_3 = 10 & \text{or } x_3 = 1 \end{array}\right\} \text{impossible!}$$

So we can't go back-and-forth between solutions (x_1, x_2) of set A and (x_1, x_2, x_3) of set B; the sets are not equivalent.

2. Problem set 3.1, question 3 (page 60)

- part (a)

$$\text{Minimize} \quad -3x_1 + 2x_2$$

$$\begin{aligned} \text{subject to} \quad & 5x_1 + 2x_2 - 3x_3 + x_4 + x_5 && = 7 \\ & \quad\; 3x_2 - 4x_3 \qquad\qquad + x_6 && = 6 \\ & x_1 \qquad + x_3 - x_4 \qquad\qquad - x_7 && = 11 \end{aligned}$$

$$x_i \geq 0, \; i = 1, \ldots, 7$$

- part (e)

$$\text{Minimize} \quad 6x_1' - 6x_1'' + x_2$$

$$\begin{aligned} \text{subject to} \quad & -5x_1' + 5x_1'' + 8x_2 + x_3 && = 80 \\ & x_1' - x_1'' \quad + 2x_2 \qquad - x_4 && = 4 \\ & x_1' - x_1'' \qquad\qquad\qquad + x_5 && = 10 \end{aligned}$$

NB: this encodes $x_1 \leq 10$!

$$x_1', x_1'', x_2, x_3, x_4, x_5 \geq 0$$

- part (g)

Encode $-x_1 - x_2 + x_4 \geq \max\{7x_1 + 2x_2, 5x_2 + x_3 + x_4\}$

by pair of constraints:
$$-x_1 - x_2 + x_4 \geq 7x_1 + 2x_2$$
$$-x_1 - x_2 + x_4 \geq 5x_2 + x_3 + x_4$$

Standard form: Minimize $-x_1 - 6x_2 - 12x_3$

$$\begin{aligned} \text{subject to:} \quad & -8x_1 - 3x_2 \qquad + x_4 - x_5 && = 0 \\ & -x_1 - 6x_2 - x_3 - x_4 \qquad - x_6 && = 0 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

3. Problem set 3.1, question 4 (page 61)

- part (a) Setting $x_1 = x_2 = x_4 = 0$ yields system

$$-3x_3 \leq 7 \quad (\text{or } x_3 \geq -2\tfrac{1}{3})$$
$$-4x_3 \leq 6 \quad (\text{or } x_3 \geq -1\tfrac{1}{2})$$
$$x_3 \geq 11$$
$$x_3 \geq 0$$

The only simultaneous solutions are those $x_3 \geq 11$, so the set of all feasible soln's is the set

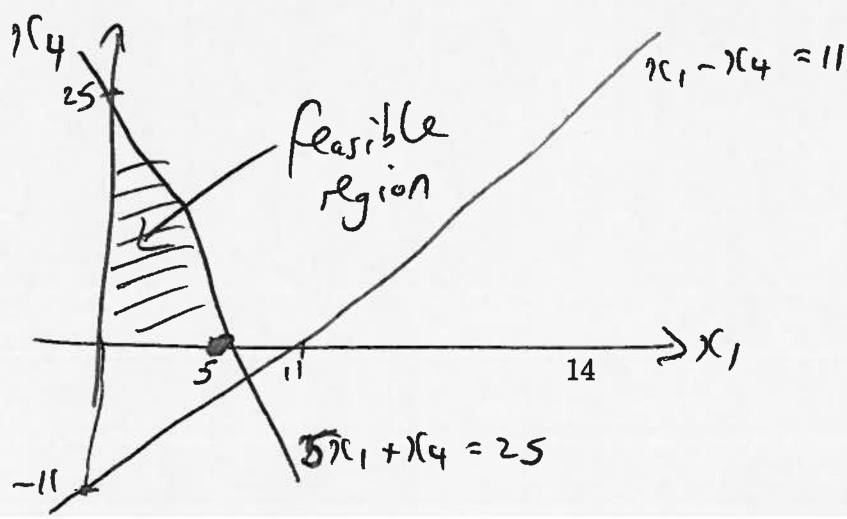
$$\{(0,0,x_3,0) : x_3 \geq 11\}$$

- part (b) [for this part, illustrate the set of feasible points on a graph]

setting $x_2 = 0$, $x_3 = 6$ yields system

$$5x_1 - 18 + x_4 \leq 7 \quad \text{or} \quad 5x_1 + x_4 \leq 25$$
$$-24 \leq 6 \quad [\text{always true}]$$
$$x_1 + 6 - x_4 \leq 11 \quad \text{or} \quad x_1 - x_4 \leq 11$$
$$x_1, x_4 \geq 0$$

Graphical illustration of feasible set:



4. Problem set 3.1, question 7 (page 62) [here's what this question is asking: show that for any feasible point for Problem A, there's a corresponding feasible point for Problem B with the same objective value, and vice-versa. This problem shows that if you have a bunch unrestricted variables, you can put them into standard form simultaneously with the addition of a single new variable.]

Suppose (x_1, x_2, x_3, x_4) is feasible for problem A

Choose x_2', x_3', x_4', x_0 such that

i) $x_2', x_3', x_4', x_0 \geq 0$

ii) $x_2' - x_0 = x_2$, $x_3' - x_0 = x_3$, $x_4' - x_0 = x_4$

[e.g. if $x_2 = -2$, $x_3 = 7$, $x_4 = 3$, then $x_2' = 2$, $x_3' = 11$, $x_4' = 7$, $x_0 = 4$ works]

[In general, letting $x_0 = 0$ and $x_2' = x_2$ etc if $x_2, x_3, x_4 \geq 0$; and $x_0 = -\text{minimum}\{x_2, x_3, x_4\}$ if some of them are negative; then $x_2' = x_2 - x_0$, etc, works]

Easy check: $(x_1, x_2', x_3', x_4', x_0)$ is feasible for problem B, and objective of B at this point is same as objective of A at (x_1, x_2, x_3, x_4)

Other direction: Suppose $(x_1, x_2', x_3', x_4', x_0)$ is feasible for problem B.

Set $x_2 = x_2' - x_0$, $x_3 = x_3' - x_0$, $x_4 = x_4' - x_0$.

Easy check: (x_1, x_2, x_3, x_4) satisfies all constraints of problem A, so is feasible for problem A; and objective of A at this point is same as objective of B at $(x_1, x_2', x_3', x_4', x_0)$

5. Problem set 3.2, question 1 (page 70)

- part (a)

$$\begin{array}{rrrl} & 3x_2 & -3x_3 & = 15 \\ x_1 & +x_2 & +x_3 & = 0 \\ 3x_1 & +5x_2 & +3x_3 & = 4 \end{array} \rightarrow \begin{array}{rrrl} & 3x_2 & -3x_3 & = 15 \\ x_1 & +x_2 & +x_3 & = 0 \\ & 2x_2 & & = 4 \end{array}$$

$$\rightarrow \begin{array}{rrrl} & & -3x_3 & = 9 \\ x_1 & & +x_3 & = -2 \\ & x_2 & & = 2 \end{array} \rightarrow \begin{array}{rrrl} & & x_3 & = -3 \\ x_1 & & & = 1 \\ & x_2 & & = 2 \end{array}$$

Solution : $x_1 = 1, \; x_2 = 2, \; x_3 = -3$

- part (b)

$$\begin{array}{rrrl} 3x_1 & +2x_2 & -7x_3 & = 1 \\ x_1 & -5x_2 & -6x_3 & = -4 \end{array} \rightarrow \begin{array}{rrrl} & 17x_2 & +11x_3 & = 13 \\ x_1 & -5x_2 & -6x_3 & = -4 \end{array}$$

$$\rightarrow \begin{array}{rrrl} & x_2 & +\frac{11}{17}x_3 & = \frac{13}{17} \\ x_1 & & -\frac{47}{17}x_3 & = -\frac{3}{17} \end{array}$$

So solution is
$$x_1 = -\frac{3}{17} + \frac{47}{17}x_3$$
$$x_2 = \frac{13}{17} - \frac{11}{17}x_3$$
(x_3 a free variable)

[NB : Could also have solved x_1, x_3 in terms of x_2; x_2, x_3 in terms of x_1.]

6. Problem set 3.2, question 2 (page 71)

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ -2x_1 + x_2 + 5x_3 &= 2 \\ 3x_2 - x_3 &= 16 \end{aligned} \longrightarrow \begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ 3x_2 - x_3 &= 16 \\ 3x_2 - x_3 &= 16 \end{aligned}$$

This shows that

$$\underbrace{\text{Constraint } 3 = \text{Constraint } 2 + 2[\text{Constraint } 1]}_{\text{redundancy!}}$$

Pivoting once more, get

$$\begin{aligned} x_1 \quad - 2\tfrac{2}{3}x_3 &= 1\tfrac{2}{3} \\ x_2 \quad -\frac{x_3}{3} &= 5\tfrac{1}{3} \\ 0 &= 0 \end{aligned}$$

So there is a whole <u>line</u> of solutions;

$$\left[1\tfrac{2}{3} + 2\tfrac{2}{3}x_3,\ 5\tfrac{1}{3} + \tfrac{1}{3}x_3,\ x_3\right]$$

<u>IF</u> there was a canonical form, there would be a <u>unique</u> soln; so <u>no</u>, there is no canonical form in this case.

7. Problem set 3.2, question 3 (page 71)

a) $$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x_1 + 2x_2 &= 3 \\ \underbrace{0 = 1}_{} \end{aligned}$$

Impossible! So system is inconsistent

b) $$\begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ -2x_1 + x_2 + 5x_3 &= 2 \\ 3x_2 - x_3 &= 15 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ 3x_2 - x_3 &= 16 \\ 3x_2 - x_3 &= 15 \end{aligned}$$

$$\longrightarrow \quad \begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ x_2 - \frac{x_3}{3} &= 16/3 \\ \underbrace{0 = -1}_{} \end{aligned}$$

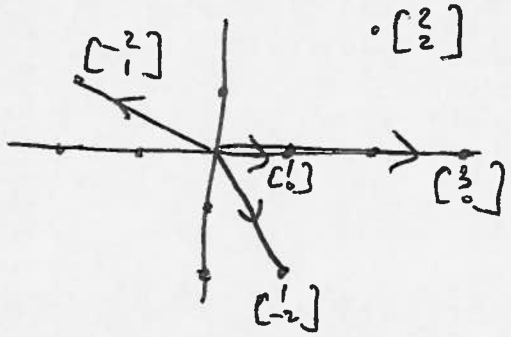
Impossible! So system is inconsistent.

No canonical form in either case; if we had a canonical form, we would have at least one soln to the system (the basic soln)

8. Problem set 3.2, question 6 (page 72)

- part (a) System is $x_1\begin{bmatrix}1\\-2\end{bmatrix} + x_2\begin{bmatrix}1\\0\end{bmatrix} + x_3\begin{bmatrix}-2\\1\end{bmatrix} + x_4\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

in vector notation



From picture, we see that $\begin{bmatrix}2\\2\end{bmatrix}$ is in positive cone of $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}$, and of $\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}$ but is <u>not</u> in positive cone of any other pair. Hence there are only two possible pairs of basic variables: x_2, x_3 and x_3, x_4

- part (b)

Basic vars x_2, x_3 : Solve $x_2\begin{bmatrix}1\\0\end{bmatrix} + x_3\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$; $(x_1 = x_4 = 0)$

$x_1 = 0$
$x_2 = 6$
$x_3 = 2$
$x_4 = 0$

Basic vars x_3, x_4 : Solve $x_3\begin{bmatrix}-2\\1\end{bmatrix} + x_4\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$; $(x_1 = x_2 = 0)$

$x_1 = 0$
$x_2 = 0$
$x_3 = 2$
$x_4 = 2$

- part (c)

objective function is positive linear combination of non-negative quantities, so must be ≥ 0

- part (d)

Basic feasible soln $(0, 6, 2, 0) \to$ obj $= 18$

" " " $(0, 0, 2, 2) \to$ obj $= 8$

Assuming obj reaches min at some basic feasible soln, minimum is 8

9. Problem set 3.2, question 9 (page 72)

Original system: n vars $x_1, \dots, x_n$,

m linear equalities
$$\left.\begin{aligned} f_1(x_1, \dots, x_n) &= b_1 \\ f_2(x_1, \dots, x_n) &= b_2, \\ &\dots, \\ f_m(x_1, x_2, \dots, x_n) &= b_m. \end{aligned}\right\} \text{System } A$$

After pivoting: there's $a \neq 0$ and $a_2, \dots, a_m$, and system is

$$\text{System } B \quad \left\{\begin{aligned} a f_1(x_1, \dots, x_n) &= a b_1, \\ f_2(x_1, \dots, x_n) - a_2 f_1(x_1, \dots, x_n) &= b_2 - a_2 b_1, \\ &\vdots \\ f_m(x_1, \dots, x_n) - a_m f_1(x_1, \dots, x_n) &= b_m - a_m b_1 \end{aligned}\right.$$

For convenience, write $\vec{x}$ for $(x_1, \dots, x_n)$.

One direction: Suppose $\vec{x}^* = (x_1^*, \dots, x_n^*)$ satisfies all constraints of system A.

Want to show that it satisfies all constraints of system B.

Since $f_1(\vec{x}^*) = b_1$, have $a f_1(\vec{x}^*) = a b_1$, so ✓

Since also $f_2(\vec{x}^*) = b_2$, have

$f_2(\vec{x}^*) - a_2 f(\vec{x}^*) = b_2 - a_2 b_1$ ✓, same for $3, \dots, m$.

Other direction: suppose $\vec{x}^* = (x_1^*, \dots, x_n^*)$ satisfies all constraints of system B

Want to show that it satisfies all constraints of system A

Since $a f_1(\vec{x}^*) = a b_1$, and $a \neq 0$, have $f_1(\vec{x}^*) = b_1$ ✓

Since $f_2(\vec{x}^*) - a_2 f_1(\vec{x}^*) = b_2 - a_2 b_1$ using $f_1(\vec{x}^*) = b_1$,

get $f_2(\vec{x}^*) - a_2 b_1 = b_2 - a_2 b_1$, so $f_2(\vec{x}^*) = b_2$.

Same for $3, \dots, m$.

So $\vec{x}^*$ satisfies all constraints of $A \iff$ it satisfies all constraints of B ⊗