5 Homework 3 (due Sept. 17) Name: SOLUTIONS

The purpose of this homework is to explore the simplex algorithm for solving a linear programming problem in standard form.

Reading: Sections 3.3 and 3.4

1. Problem set 3.3, question 1 (page 76)

   - part (a)

$$x_1 = 8 - 2x_4$$
$$x_2 = 6 - 3x_4$$
$$x_3 = 18 - 6x_4$$

(set of solutions is $(8-2x_4,\ 6-3x_4,\ 18-6x_4,\ x_4)$, with x_4 ranging over all values that keep all these terms positive)

   - part (b)

Need: $x_4 \geq 0$

$8 - 2x_4 \geq 0$ or $x_4 \leq 4$

$6 - 3x_4 \geq 0$ of $x_4 \leq 2$

$18 - 6x_4 \leq 0$ or $x_4 \leq 3$

For all four to hold, need $0 \leq x_4 \leq 2$

   - part (c)

At $x_4 = 2$, $\underline{x_2 = 0}$

- part (d)

x_2 should be extracted from basis

(we should pivot on the $3x_4$ term)

- part (e)

$$\begin{aligned} x_1 \qquad\qquad + 2x_4 &= 8 \\ x_2 \qquad + 3x_4 &= 6 \\ x_3 + 6x_4 &= 18 \end{aligned} \quad \rightsquigarrow \quad \underbrace{\begin{aligned} x_1 - \tfrac{2}{3}x_2 \qquad\qquad &= 4 \\ \tfrac{x_2}{3} \qquad + x_4 &= 2 \\ + x_3 \qquad &= 6 \end{aligned}}$$

Canonical form $\bar{w}$
x_1, x_3, x_4 basic,
as desired

- part (f)

Ratios are $\frac{8}{2} = 4$

$\frac{6}{3} = 2$ ← this is the smallest ~~term~~; it determined the pivot location

$\frac{18}{6} = 3$

2. Problem set 3.3, question 2 (page 76)

- part (a)

$$\text{Min } z, \quad \begin{aligned} x_1 + x_2 + 4x_3 + 7x_4 &= z \\ x_1 + x_2 + 5x_3 + 2x_4 &= 8 \\ 2x_1 + x_2 + 8x_3 &= 14 \end{aligned} \quad \rightsquigarrow \quad \text{Min } z \quad \begin{aligned} -x_3 + 5x_4 &= z - 8 \\ x_1 + x_2 + 5x_3 + 2x_4 &= 8 \\ -x_2 - 2x_3 - 4x_4 &= -2 \end{aligned}$$

$\rightsquigarrow$ Min z subject to:

$$\left.\begin{aligned} -x_3 + 5x_4 &= -8 + z \\ x_1 \qquad + 3x_3 - 2x_4 &= 6 \\ x_2 + 2x_3 + 4x_4 &= 2 \end{aligned}\right\} \text{ as claimed}$$

- part (b)

Key point

Right now we've expressed

$$(*) \rightarrow \boxed{z = -x_3 + 5x_4 + 8,}$$

and we have feasible point with $x_3 = x_4 = 0$

Because coefficient of x_3 is negative in current expression for z, if we increase x_3 that makes rhs of $(*)$ smaller, so it makes lhs of $(*)$ smaller, too; that is, it decreases the objective value.

- part (c)

Because smaller of $\frac{6}{3}, \frac{2}{2}$ is $\frac{2}{2}$, we should pivot on circled $2x_3$ to remove x_2 from basis. Get:

$$\left.\begin{array}{lllll} & \frac{x_2}{2} & & +7x_4 & = -7+z \\ x_1 & -\frac{3x_2}{2} & & -8x_4 & = 3 \\ & \frac{x_2}{2} & +\, x_3 & +2x_4 & = 1 \end{array}\right\}$$

Minimize z subject to these constraints

We're at bfs $(3, 0, 1, 0)$

line 2 line 3

x_2, x_4 not-basic;

Objective $z = 7 + \frac{x_2}{2} + 7x_4$; is 7 at current bfs,

Can't get any lower since

$$7 + \frac{x_2}{2} + 7x_4 \geq 7 \text{ always when } x_2, x_4 \geq 0.$$

3. Problem set 3.3, question 3 (page 77)

Solved using LP Assistant, both with x_1 entering first and x_4 entering first.

In either case, optimum is -2, achieved at $(2, 9, 0, 0)$

Q1_enter_x1_first

Tableau

Mode: Edit, Pivot
Algorithm: Simplex, Dual Simplex
Display: 1/2, 1.0, 1.00, 1.000
Ratio

Basis	X1	X2	X3	X4	X5	RHS
x_3	1	0	1	6	3	2
x_2	-3	1	0	3	1	3
	-1	0	0	-2	1	0
x_1	1	0	1	6	3	2
x_2	0	1	3	21	10	9
	0	0	1	4	4	2

Q1_enter_x4_first

Tableau

Mode: Edit, Pivot
Algorithm: Simplex, Dual Simplex
Display: 1/2, 1.0, 1.00, 1.000
Ratio

Basis	X1	X2	X3	X4	X5	RHS
x_3	1	0	1	6	3	2
x_2	-3	1	0	3	1	3
	-1 -1	0	0	-2	1	0
x_4	$\frac{1}{6}$	0	$\frac{1}{6}$	1	$\frac{1}{2}$	$\frac{1}{3}$
x_2	$-\frac{7}{2}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	2
	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	2	$\frac{2}{3}$
x_1	1	0	1	6	3	2
x_2	0	1	3	21	10	9
	0	0	1	4	4	2

4. Problem set 3.3, question 4 (page 77)

- part (a)

Min z s.t.

$$\begin{aligned} x_2 - 6x_3 - 2x_4 &= 6 \\ x_1 \quad + 2x_3 - x_4 &= 5 \\ 4x_3 - 6x_4 &= z \end{aligned}$$

After pivot:

$$\begin{aligned} \frac{x_2}{2} - 3x_3 + x_4 &= 3 \\ x_1 + \frac{x_2}{2} - x_3 \quad &= 8 \\ 3x_2 - 14x_3 \quad &= z + 18 \end{aligned}$$

What's new: column above only negative entry in objective row has only <u>negative</u> coefficients!

- part (b)

Currently $z = -18 + 3x_2 - 14x_3$, at bfs $(8,0,0,3)$,

Suppose we increase x_3 from 0 to M (large number), keeping $x_2 = 0$.

From $x_1 - x_3 = 8$, learn that we should change x_1 to $M+8$

From $x_4 - 3x_3 = 3$, " " " " " x_4 to $3M+3$

Have bFs $(M+8, 0, M, 3M+3)$ (feasible because $M \geq 0$ so $M+8, 3M+3 \geq 0$)

with obj value $z = -18 + 3 \cdot 0 - 14M = -14M - 18$

By making M arbitrarily large, can make $-14M-18$ arbitrarily small, so problem is <u>not</u> bounded from below.

5. Problem set 3.4, question 2 (page 83)

- part (a)

Current basic feasible solution

$x_1 = 5$, $x_2 = 10$, $x_3 = 0$, $x_4 = 0$

$$z = \underbrace{x_3 + x_4}_{\text{expressed in terms of non-basic variables.}} = 0$$

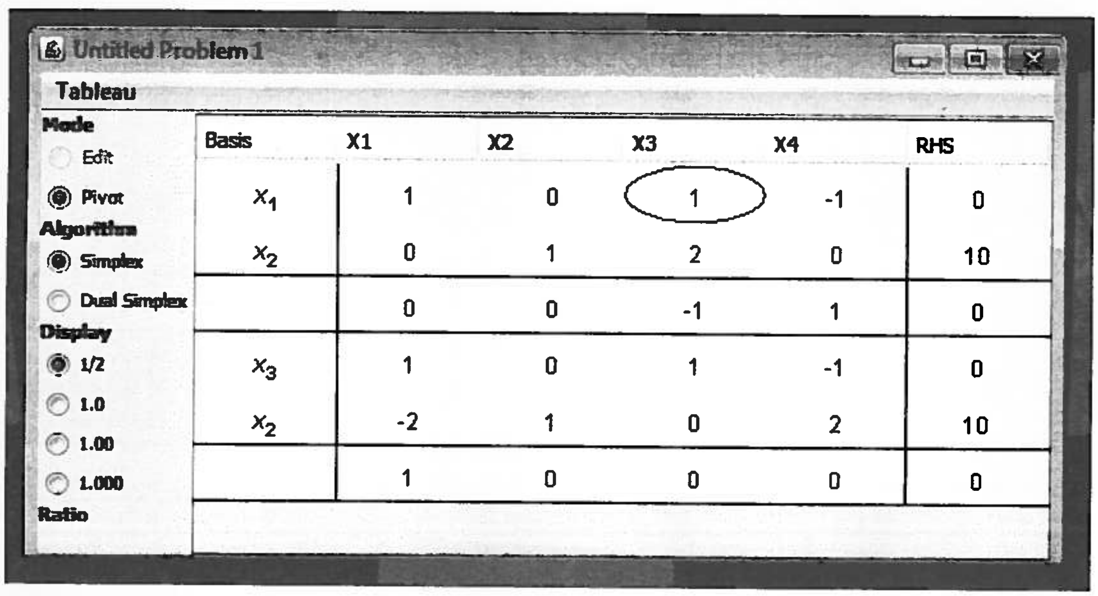
Optimality criterion applies immediately!

Optimum is $\underline{0}$

- part (f)

Solved (quite quickly :)) via LP Assistant;
optimum is 0 achieved at

$x_1 = 0 \quad x_2 = 10 \quad x_3 = 0 \quad x_4 = 0$



Basis	X1	X2	X3	X4	RHS
x_1	1	0	1	-1	0
x_2	0	1	2	0	10
	0	0	-1	1	0
x_3	1	0	1	-1	0
x_2	-2	1	0	2	10
	1	0	0	0	0

6. Problem set 3.4, question 6 (page 84)

Here's the relevant portion of the tableau:

Basis	x_1	x_2	x_3	$\cdots x_7$	$\cdots$	x_{10}	rhs
x_1	1			a_{17}			0
				a_{27}			b_2
				$\vdots$			$\vdots$
				a_{57}			b_5
obj	0			c_5			$\sim$

I'm assuming (wlog) that we've decided to enter x_7 into the basis (so c_5 is negative, and, since we are actually pivoting, not all of $a_{17}, a_{27}, \ldots, a_{57} \leq 0$)

Case i: If $a_{17} > 0$, then the x_1 row imposes constraint $x_1 + a_{17}x_7 = 0$, which says that x_7 can only be increased to 0 (from 0) before x_1 hits 0.

In this case, x_7 goes from 0 (non-basic) to 0 (basic)
x_1 goes from 0 (and basic) to 0 (and non-basic)

There is <u>no change</u> to the values of the variables or objective at basic feasible solution; the only change is the cosmetic one that the names of the basic variables have changed.

Case ii: If $a_{17} \leq 0$, then the ratio $\frac{0}{a_{17}}$ isn't considered during the pivoting, and without knowing values of b_2, a_{27}, etc, we can't say anything. But, since question explicitly says that x_1 is departing the basis, I don't think that this case can arise.