MATH 30210, FALL 2014, HOMEWORK 6 SOLNS

1) Suppose primal problem P is a maximization problem, and that it is unbounded.

Claim: Dual problem D has no feasible point

Proof: Suppose D has a feasible point with objective value t.

Because P is unbounded, P has a feasible point with objective value $t+1$

Since $t+1 > t$, this contradicts the weak duality theorem (Thm 4.4.1) that says that any feasible objective for P is at most as large as any feasible objective for D

So D has no feasible point.

If primal problem P is a minimization problem, and is unbounded, then almost exactly the same proof shows that D has no feasible points.

2) Problem constraints are equivalent to:

$$x_1 - x_2 \le 1 \text{ and}$$
$$x_1 - x_2 \ge 2;$$

clearly there is no pair (x_1, x_2) that simultaneously satisfies both constraints, so problem has no feasible points

Dual problem is to Minimize $2y_1 - 2y_2$

subject to
$$y_1 - y_2 \ge 1$$
$$-y_1 + y_2 \ge 0$$
$$y_1, y_2 \ge 0$$

Dual constraints are equivalent to:

$$y_1 - y_2 \ge 1 \text{ and}$$
$$y_1 - y_2 \le 0;$$

Clearly no (y_1, y_2) simultaneously satisfies both constraints, so dual problem has no feasible points.

3) first, check feasibility of $(0, 5\frac{2}{3}, 8\frac{1}{3}, \frac{1}{3})$:

$2(5\frac{2}{3}) - 8\frac{1}{3} - 3(\frac{1}{3}) = 2 \geq 2$ ✓

$5(5\frac{2}{3}) - 2(8\frac{1}{3}) + (\frac{1}{3}) = 12 \geq 12$ ✓

$-4(5\frac{2}{3}) + 3(8\frac{1}{3}) + 5(\frac{1}{3}) = 4 \geq 4$ ✓

and each of $0, 5\frac{2}{3}, 8\frac{1}{3}, \frac{1}{3} \geq 0$

So $(0, 5\frac{2}{3}, 8\frac{1}{3}, \frac{1}{3})$ is a feasible point.

Objective value: $11(5\frac{2}{3}) - 3(8\frac{1}{3}) - (\frac{1}{3}) = 37$

Next, check feasibility of $(3\frac{1}{2}, 2, 1\frac{1}{2})$ for dual:

$2(3\frac{1}{2}) - (2) + (1\frac{1}{2}) = 6\frac{1}{2} \leq 7$ ✓

$2(3\frac{1}{2}) + 5(2) - 4(1\frac{1}{2}) = 11 \leq 11$ ✓

$-(3\frac{1}{2}) - 2(2) + 3(1\frac{1}{2}) = -3\frac{1}{2} \leq -3$ ✓

$-3(3\frac{1}{2}) + (2) + 5(1\frac{1}{2}) = -1 \leq -1$ ✓

and each of $3\frac{1}{2}, 2, 1\frac{1}{2} \geq 0$

So $(3\frac{1}{2}, 2, 1\frac{1}{2})$ is a feasible point for dual

Objective value: $2(3\frac{1}{2}) + 12(2) + 4(1\frac{1}{2}) = 37$

Since we have found feasible points for primal and dual w/ same objective, by weak duality (specifically Cor 4.4.2) these two points are optimal for their respective problems

4) a) $4(9) - 5(2) + 3(2) = 32$

$3(9) + 6(2) - (2) = 37 \geq 3$

So $(9,0,2,2)$ is feasible

~~$4(4) + 8(1) - 5(-1) + 3(1) = 32$~~

~~$3(4) - 2(1) + 6(-1) - 1(1) = 3 \geq 3$~~

~~So $(4,1,-1,1)$ is feasible~~

$-1 \not\geq 0$, so $(4,1,-1,1)$ is not feasible

$4(5) + 8(1) - 5(1) + 3(3) = 32$

$3(5) - 2(1) + 6(1) - 1(3) = 16 \geq 3$

So $(5,1,1,3)$ is feasible

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b) At $(9,0,2,2)$ $z = 13(9) + 12(2) + 8(2)$
$= 157$

~~At $(4,1,-1,1)$ $z = 13(4) + 15(1) + 12(-1) + 8(1)$~~
~~$= 63$~~

At $(5,1,1,3)$, $z = 13(5) + 15(1) + 12(1) + 8(3)$
$= 116$

c) Max $32y_1 + 3y_2$ subject to

$4y_1 + 3y_2 \leq 13$

$8y_1 - 2y_2 \leq 15$

$-5y_1 + 6y_2 \leq 12$

$3y_1 - y_2 \leq 8$

y_1 unrestricted,

$y_2 \geq 0$.

d) Check $(-1,1)$: $-4+3 \le 13$

$-8-2 \le 15$

$5+6 \le 12$

$-3-1 \le 8$

and $1\ (y_2) \ge 0$ ✓

$(-1,1)$ feasible

Check $(0,2)$: $6 \le 13$

$-4 \le 15$

$12 \le 12$

$-2 \le 12$

and $2 \ge 0$

$(0,2)$ feasible

Check $(1,3)$: $4+9 \le 13$

$8-6 \le 15$

$-5+18 = 13 \not\le 12$

So $(1,3)$ is not feasible

e) At $(-1,1)$, $w = 32(-1) + 3(1) = -29$

At $(0,2)$, $w = 32(0) + 3(2) = 6$

f) We can say that the minimum lies somewhere between 6 [the largest known dual feasible value] and 116 [the smallest known primal feasible value]

5) Dual: Maximize $2y_1 + 4y_2 - y_3 + 0y_4 - 2y_5$

subject to:

$$
\begin{aligned}
3y_1 + y_2 \qquad\qquad - 2y_4 + 4y_5 &\le 8 \\
2y_1 - y_2 + 2y_3 + 3y_4 \qquad\quad &\le 13 \\
y_1 + 2y_2 + 2y_3 \qquad\qquad - y_5 &\le 20
\end{aligned}
$$

$y_1, y_2, y_3, y_4, y_5 \ge 0$

Initial simplex tableau:

basis	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	rhs
						slack vars			
y_6	3	1	0	-2	4	1	0	0	8
y_7	2	-1	2	3	0	0	1	0	13
y_8	1	2	2	0	-1	0	0	1	20
Obj	-2	-4	1	0	2	0	0	0	0

dual is max problem:
flip objective so simplex can solve a MIN problem

Final simplex tableau:

basis	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	rhs
y_2									
y_7	IRRELEVANT								
y_4									
Obj	NONSENSE					0	0	2	40

From final tableau, we see that

$x_1 = 0$
$x_2 = 0$
$x_3 = 2$
(objective row coefficients of slack variables associated with constraints 1, 2, 3 of primal)

is an optimal solution for the primal, with optimum objective value $\underline{\underline{40}}$

6) a) Dual: Maximize $13y_1 - 2y_2 + 5y_3$

subject to $2y_1 - 6y_2 + 7y_3 \leq 100$

$y_1 + 9y_2 - 8y_3 \leq 150$

$y_1, y_2, y_3 \geq 0$

Solution to dual: $y_1 = 75$ [negative of obj row coefficient of artificial variable associated with constraint 1]

$y_2 = \frac{25}{3}$ [obj row coefficient of slack variable associated with constraint 2]

$y_3 = 0$ [neg of obj row coeff. of arti. var. associated w constraint 3]

Check feasibility: $2(75) - 6\left(\frac{25}{3}\right) = 100$ ✓

$75 + 9\left(\frac{25}{3}\right) = 150$ ✓

$75, \frac{25}{3}, 0 \geq 0$ ✓

Check optimality: $13(75) - 2\left(\frac{25}{3}\right) = \frac{2875}{3}$,

which is exactly optimum (from tableau) of the original minimization problem; so by duality thm, $\left(75, \frac{25}{3}, 0\right)$ is optimal for dual.

c) Dual: Max $y_1 - 6y_2 + 3y_3$

subject to
$$-3y_2 + 2y_3 \le -2$$
$$2y_1 + y_2 - 4y_3 \le 5$$
$$5y_1 + y_2 + y_3 \le 9$$
$$y_1, y_2 \ge 0, \ y_3 \text{ unrestricted}$$

Dual solution: $y_1 = \frac{7}{4}$

$y_2 = \frac{1}{2}$

$y_3 = -\frac{1}{4}$

Check feasibility: $-3(\frac{1}{2}) + 2(-\frac{1}{4}) = -2$

$2(\frac{7}{4}) + (\frac{1}{2}) - 4(-\frac{1}{4}) = 5$

$5(\frac{7}{4}) + (\frac{1}{2}) + (-\frac{1}{4}) = 9$

$\frac{7}{4}, \frac{1}{2} \ge 0$

Check optimality: $(\frac{7}{4}) - 6(\frac{1}{2}) + 3(-\frac{1}{4}) = -2$,

which is optimal value of primal

e) Dual: Minimize $90y_1 - 42y_2$

subject to
$$6y_1 + y_2 \geq 10$$
$$-7y_1 \geq -12$$
$$8y_1 - 3y_2 \geq 11$$
y_1 unrestricted, $y_2 \geq 0$

Dual solution: $y_1 = \frac{12}{7}$

$y_2 = \frac{19}{21}$

Check feasibility: $6\left(\frac{12}{7}\right) + \left(\frac{19}{21}\right) = \frac{235}{21} \geq 10$ ✓

$-7\left(\frac{12}{7}\right) = -12 \geq -12$ ✓

$8\left(\frac{12}{7}\right) - 3\left(\frac{19}{21}\right) = 11$ ✓

$\frac{19}{21} \geq 0$ ✓

Check optimality: $90\left(\frac{12}{7}\right) - 42\left(\frac{19}{21}\right) = \frac{814}{7}$,

which is the optimum value of the primal ✓