11 Homework 7 (due Oct. 31) Name: SOLUTIONS

The purpose of this homework is to explore the use of integer constraints in linear programming problems, and in particular the use of auxiliary variables.

Reading: Sections 6.1 and 6.2.

1. Problem set 6.2, question 5 (page 224). [Just the first and third figures!]

- Figure 6.3 Note: I'm using (x, y) for (x_1, x_2)

Feasible region described by:

$x \geq 0$ AND $y \geq 0$ AND $[\underbrace{5x+4y \leq 20}_{①}$ OR $\underbrace{3x+8y \leq 24}_{②}]$

Use auxiliary variables w_1, w_2, to "switch on" ①, ②.

① : $5x+4y \leq 20 + [1-w_1] \times 40$

② : $3x+8y \leq 24 + [1-w_2] \times 40$

[biggest that $5x+4y$ could possibly be in feasible region is $5[8]+4[5] = 60$

biggest that $3x+8y$ can be is $3[8]+8[5] = 64$]

Use $w_1 + w_2 \geq 1$ to encode the "OR" statement.

New system:

$$\begin{aligned}
5x+4y &\leq 20 + [1-w_1]40 \\
3x+8y &\leq 24 + [1-w_2]40 \\
w_1 + w_2 &\geq 1 \\
w_1 &\leq 1 \\
w_2 &\leq 1
\end{aligned}$$

$x, y, w_1, w_2 \geq 0$, $\underline{w_1, w_2 \text{ integers}}$

- Figure 6.5 Note: I'm using (x, y) for (x_1, x_2)

Feasible region is: $x \geq 0$ AND $y \geq 0$ AND

$$\left[2x + 5y \leq 10 \text{ OR } \left\{ -x + y \leq 0 \text{ AND } x \leq 3 \right\} \right]$$

Use auxiliary variable w_1 to "switch on" $2x + 5y \leq 10$:

Constraint $2x + 5y \leq 10 + M[1 - w_1]$

Choose M large enough that when $w_1 = 0$, don't cut off any feasible points; ie, M large enough so that $10 + M$ is bigger than $2x + 5y$ on feasible region.

Biggest feasible value of x: 5 so $2x + 5y \leq 25$; take $M = 15$

" " " " y: 3

Use aux. var w_2 to "switch on" both $-x + y \leq 0$ And $x \leq 3$:

$-x + y \leq 0 + 3[1 - w_2]$ $[-x + y \leq y \leq 3$ on feasible region$]$

$x \leq 3 + 2[1 - w_2]$ $[x \leq 5$ on feasible region$]$

Add $w_1 + w_2 \geq 1$ to deal with the "OR"

Full new set of constraints:

$$2x + 5y \leq 10 + 15[1 - w_1]$$
$$-x + y \leq 3[1 - w_2]$$
$$x \leq 3 + 2[1 - w_2]$$
$$w_1 + w_2 \geq 1$$
$$x, y, w_1, w_2 \geq 0,$$

w_1, w_2 both integers.

2. Problem set 6.2, question 6 (page 225). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

$x_1 = \begin{cases} 1 & \text{if development project investment made} \\ 0 & \text{o'wise} \end{cases}$

x_2 = same for construction project

x_3 = # units of stock purchased (portfolios)

x_4 = # " " " " (individual)

objective: Maximize profit

$$P = (390000)(.067)x_1 + (220000)(.065)x_2 + (25000)(.063)x_3 + (1300)(.057)x_4$$

subject to:

$$390000x_1 + 220000x_2 + 25000x_3 + 1300x_4 \le 500000$$

$$x_1 \le 1$$

$$x_2 \le 1$$

$$x_1, x_2, x_3, x_4 \ge 0$$

All x_i integers.

3. Problem set 6.2, question 9 (page 225). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

x_1 = # 10-week state periods

x_2 = # 6-week county periods

x_3 = # 3-week periods (land dev)

x_4 = # 1-week parking lot periods

y = aux. var $\rightarrow y = \begin{cases} 0, & \text{if } x_1 + x_2 = 0 \\ 1, & \text{if } x_1 + x_2 \geq 1 \end{cases}$

Objective: Maximize profit

$$p = 10[3200]x_1 + 6[2900]x_2 + 3[2750]x_3 + 1[2550]x_4 - y_1(7500)$$

Subject to:

$$10x_1 + 6x_2 + 3x_3 + x_4 \leq 28$$

$$y \leq 1$$

This forces $y = 1$ if $x_1 + x_2 \geq 1$

$$\left\{ y \geq \frac{x_1 + x_2}{6} \right.$$

$x_1 + x_2 \leq 6$ always on feasible region, since $x_1 \leq 2$, $x_2 \leq 4$. Dividing by 6 ensures that rhs here never > 1

$$\left\{ y \leq x_1 + x_2 \right.$$

This forces $y = 0$ if $x_1 + x_2 = 0$, and doesn't need to be modified by dividing by a constant, since $y \leq x_1 + x_2$ never cuts off any feasible points.

$x_1, x_2, x_3, x_4, y \geq 0$, all integers.

4. Problem set 6.2, question 13 (page 227). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

Let x_i be # gallons of flavor i produced

Objective: Maximize $\sum_{i=1}^{28} c_i x_i$

subject to constraints:

① $\sum_{i=1}^{28} a_i x_i \leq M$ [NB: this says that in <u>any</u> feasible solution, $x_i \leq \frac{M}{a_i}$ → we'll need this later]

② For each i, $x_i = 0$ OR $x_i \geq U_i$

③ Each $x_i \geq 0$ [But not x_i an integer, I think? Maybe, maybe not]

To encode ② linearly: For each $i = 1, \ldots, 28$, introduce new variable y_i, $y_i \geq 0$, $y_i \leq 1$, y_i integer, and add constraints:

$y_i \geq (x_i) / (\frac{M}{a_i})$ → This forces that if $x_i > 0$, then $y_i = 1$; dividing x_i by M/a_i ensures that we are never asking y_i to exceed 1

$y_i \leq x_i/U_i$ → This forces that if $y_i = 1$, $x_i \geq U_i$, as required.

Note: from $5x_1 + 4x_2 + 2x_3 \le 300$, learn that in any feasible solution, $x_1 \le 60$, $x_2 \le 75$, $x_3 \le 150$

5. Problem set 6.2, question 15 (page 227). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it. Note also that two constraints together in the same set of braces means that BOTH of those constraints should be satisfied.]

Use auxiliary variable w_1 to "switch on" both $x_1 + x_3 \le 100$ AND $x_1 - x_2 \ge 0$:

via constraints $x_1 + x_3 \le 100 + 110[1-w_1]$ [$x_1 + x_3 \le 210$ always]

$-x_1 + x_2 \le 0 + 75[1-w_1]$

Use w_2 to "switch on" both $2x_1 - 4x_2 + 5x_3 \le 250$ AND $x_2 - 2x_3 \ge 50$,

via $2x_1 - 4x_2 + 5x_3 \le \underbrace{250 + 620[1-w_2]}_{=870 \text{ when } w_2 = 0}$;

$2x_1 - 4x_2 + 5x_3$ always $\le 2[60] + 5[150] = 870$

and $-x_2 + x_3 \le -50 + 200[1-w_2]$

Add constraint $w_1 + w_2 \ge 1$ to ensure that (at least) one of these two sets of constraints get switched on.

Full problem: Max $3x_1 + 5x_2 + 7x_3$ st

$5x_1 + 4x_2 + 2x_3 \le 300$

$x_1 + x_3 \le 100 + 110[1-w_1]$

$-x_1 + x_2 \le 75[1-w_1]$

$2x_1 - 4x_2 + 5x_3 \le 250 + 620[1-w_2]$

$-x_2 + x_3 \le -50 + 200[1-w_2]$

$w_1 \le 1$

$w_2 \le 1$

$w_1 + w_2 \ge 1$

all vars ≥ 0, w_1, w_2 integers.

Assumption: each process has to be run for an integer # of hours

6. Problem set 6.2, question 18 (page 227). [Note that you are required only to set this problem up as a linear programming problem with some integer constraints, not solve it.]

X_1 = # hours that process 1 is run for.
X_2, X_3 defined similarly.

Total labor used : $20X_1 + 12X_2 + 25X_3$ ← call this ℓ
Total materials used : $35X_1 + 12X_2 + 28X_3$ ← call this m
Total # A's produced : $40X_1 + 45X_2 + 36X_3$ ← call this a
Total # B's produced : $42X_1 + 35X_2 + 44X_3$ ← call this b

Objective : $\boxed{\text{Minimize Cost} = 12\ell + 15m}$

Subject to : labor constraint $\boxed{\ell \le 600}$

and production constraint

EITHER $a \ge 1500$ AND $b \ge 1000$ OR $a \ge 1000$ AND $b \ge 1500$

Introduce aux var w_1, $\boxed{w_1 \le 1}$, $\boxed{w_1 \ge 0}$, $\boxed{w_1 \text{ integer}}$, that "toggles" ~~both $a \ge 1500$ and $b \ge 1000$~~ between $a = 1500$ and $a = 1000$:

$\boxed{a = 1000 + 500w_1}$ (if $w_1 = 0$, $a = 1000$; if $w_1 = 1$, $a = 1500$)

Use w_1 to toggle between $b = 1000$ and $b = 1500$:

$\boxed{b = 1000 + 500[1 - w_1]}$ (if $w_1 = 0$, $b = 1500$; if $w_1 = 1$, $b = 1000$)

Full problem: Min $12\ell + 15m$ s.t.

$\ell = 20X_1 + 12X_2 + 25X_3$
$m = 35X_1 + 12X_2 + 28X_3$
$a = 40X_1 + 45X_2 + 36X_3$
$b = 42X_1 + 35X_2 + 44X_3$
$\ell \le 600$

$w_1 \le 1$
$a = 1000 + 500w_1$
$b = 1000 + 500[1 - w_1]$

all vars ≥ 0,
x_1, x_2, x_3, w_1 integers.

7. Problem set 6.2, question 1 (page 223).

- part (a)

If (x_1, x_2, y_1, y_2) satisfies (6.2-4), and $2x_1 + 9x_2 > 18$, then, since $2x_1 + 9x_2 \le 18 + (1-y_1)$, must have $y_1 = 0$. Since $y_1 + y_2 \ge 1$, must have $y_2 = 1$. Since $x_1 + x_2 - 6 \le 1 - y_2$, must have $x_1 + x_2 \le 6$

So can't have <u>both</u> $2x_1 + 9x_2 > 18$ <u>and</u> $x_1 + x_2 > 6$;

So at least one of $2x_1 + 9x_2 \le 18$ or $x_1 + x_2 \le 6$, holds.

- part (b)

At $x_1 = 3$, $x_2 = 3$, have $x_1 + x_2 \le 6$, so at least one of $2x_1 + 9x_2 \le 18$, $x_1 + x_2 \le 6$ holds at this point.

But, if there was a point $(3, 3, y_1, y_2)$ satisfying (6.2.4), we would have: $2(3) + 9(3) - 18 \le 1 - y_1$,

ie $y_1 \le -14$,

but also $y_1 \ge 0$, contradiction;

So no such point exists.

- part (c) [Note that two constraints together in the same set of braces means that BOTH of those constraints should be satisfied.]

Suppose (x_1, x_2, y_1, y_2) satisfies (6.2.4).

Three cases to consider:

Case i) $y_1 = 1, y_2 = 0$.

In this case $2x_1 + 9x_2 \leq 18$ AND $x_1 + x_2 \leq 7$ (just plug $y_1 = 1$, $y_2 = 0$ into first two inequalities of (6.2.4))

Case ii) $y_1 = 0, y_2 = 1$.

In this case, $2x_1 + 9x_2 \leq 19$ AND $x_1 + x_2 \leq 6$ (just plug in $y_1 = 0, y_2 = 1$)

Case iii) $y_1 = 1, y_2 = 1$

In this case $2x_1 + 9x_2 \leq 18$ AND $x_1 + x_2 \leq 6$ (just plug in $y_1 = y_2 = 1$),

and so both $\left\{ \begin{array}{l} 2x_1 + 9x_2 \leq 18 \\ x_1 + x_2 \leq 7 \end{array} \right\}$ and $\left\{ \begin{array}{l} 2x_1 + 9x_2 \leq 19 \\ x_1 + x_2 \leq 6 \end{array} \right\}$ hold

[Case iv, $y_1 = y_2 = 0$, violates $y_1 + y_2 \geq 1$ so we don't consider it]