12 Homework 8 (due Nov. 10) Name: SOLUTIONS

The purpose of this homework is to explore the cutting plane & branch/bound algorithms for solving integer programming problems.

Reading: Sections 6.3 and 6.4.

1. Problem set 6.3, question 1, part a) (page 236). [Just part a)]

Solving via simplex without integer constraints, final tableau has constraints

$$\tfrac{3}{4}x_1 + x_2 + \tfrac{1}{4}x_3 = 1\tfrac{1}{2}$$
$$1\tfrac{3}{4}x_1 + \tfrac{1}{4}x_3 + x_4 = 2\tfrac{1}{2}$$

(x_3 slack for first constraint, x_4 slack for second)

Whichever of the two you pick, the new constraint is

$$\tfrac{3}{4}x_1 + \tfrac{1}{4}x_3 - x_5 = \tfrac{1}{2}$$

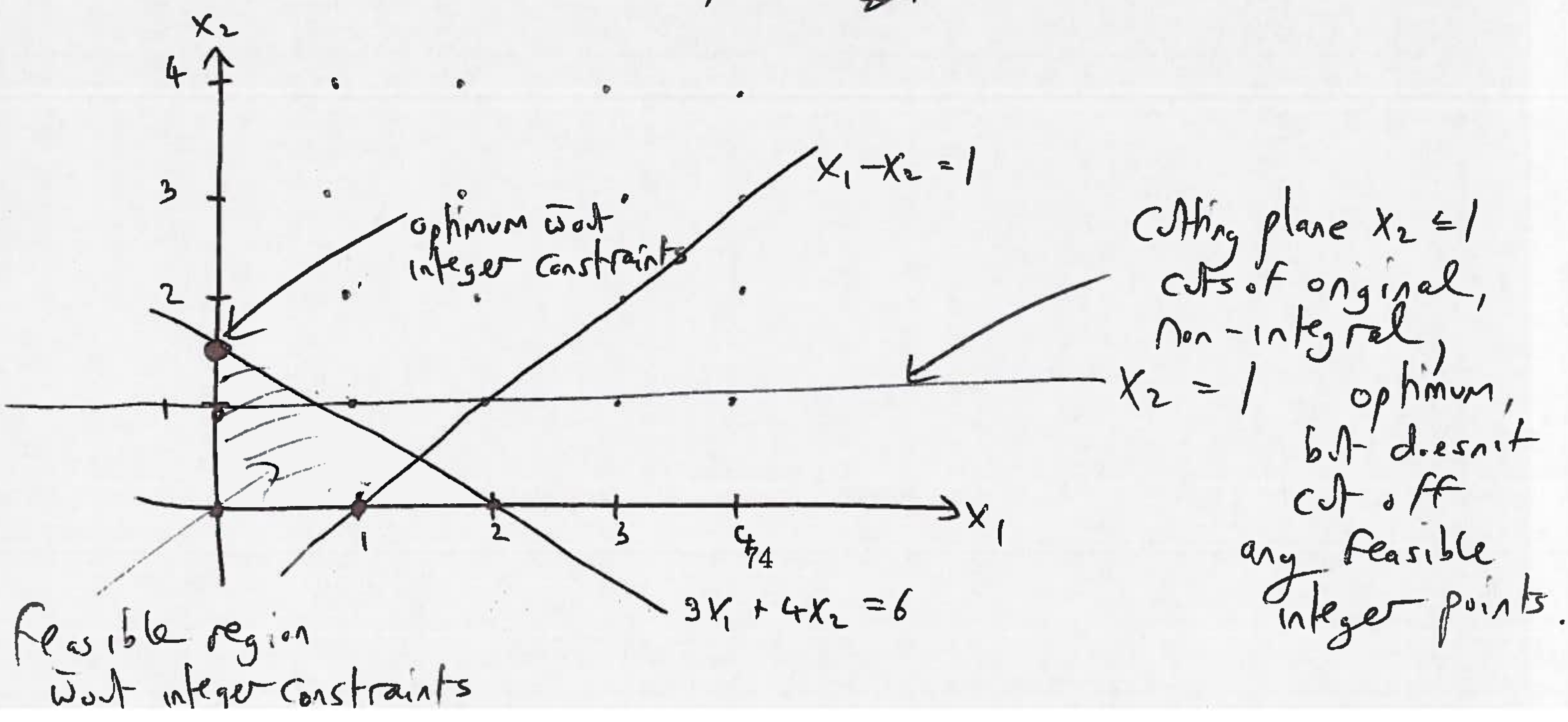
Adding this, get (via simplex) fully integral solution

$$x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 2, x_5 = 2, \quad z = -1$$

Graphically: cutting plane says $\tfrac{3}{4}x_1 + \tfrac{1}{4}x_3 \geq \tfrac{1}{2}$

subbing in $x_3 = 6 - 3x_1 - 4x_2$, this becomes

$\tfrac{3}{4}x_1 + \tfrac{1}{4}(6 - 3x_1 - 4x_2) \geq \tfrac{1}{2}$, or $x_2 \leq 1$



2. Problem set 6.3, question 2, part c) (page 236). [Just part c)]

After first iteration of simplex w/out int. constraints, have constraint $\frac{2}{3}x_2 + x_3 + \frac{1}{3}x_5 = 9\frac{1}{3}$, so add cutting plane $\frac{2}{3}x_2 + \frac{1}{3}x_5 - x_6 = \frac{1}{3}$

Running simplex w/out int. constraints on problem with this new constraint added [this iteration of simplex needs ~~[illegible]~~ artificial variables], get integral solution $x_1 = 12$, $x_2 = 0$, $x_3 = 9$, $x_4 = 0$, $x_5 = 1$, $x_6 = 0$

z (for max problem) $= 33$

So optimum for original problem is

$$\left.\begin{aligned} x_1 &= 12 \\ x_2 &= 0 \\ x_3 &= 9 \end{aligned}\right\} z = 33$$

3. Problem set 6.3, question 4 (page 237).

Possibility 1: Algorithm will run forever, each time cutting off less and less of feasible region, but never emptying it completely to allow for termination [unlikely, I think]

Possibility 2: After finitely many iterations, a problem will be reached which, when integer constraints are ignored, will have no feasible solutions, causing the algorithm to terminate [More likely, I think]

Possibilities 3, 4, 5, ... ?

4. Problem set 6.3, question 5 (page 237).

After solving initial problem w/out integer constraints, get final tableau constraints

$x_2 + \frac{x_3}{3} + \frac{x_4}{3} = \frac{1}{3} \longrightarrow$ cutting plane $\frac{x_3}{3} + \frac{x_4}{3} - x_5 = \frac{1}{3}$ ①

and $x_1 \quad - \frac{x_4}{3} = \frac{1}{3} \longrightarrow$ " " $\frac{2}{3}x_4 - x_5 = \frac{1}{3}$ ②

Adding ①: leads to having to add cutting plane $\frac{x_3}{3} - x_6 = \frac{2}{3}$; trying to solve new problem by adding this constraint, and ignoring integrality, leads to an unfeasible problem (sum of artificial variables cannot be brought down to 0)

Adding ②: leads to the addition of ~~three~~ two possible new cutting planes, $\frac{1}{3}x_3 + \frac{1}{2}x_5 - x_6 = \frac{1}{6}$ ⓐ
and $\frac{1}{2}x_5 - x_6 = \frac{1}{2}$ ⓑ

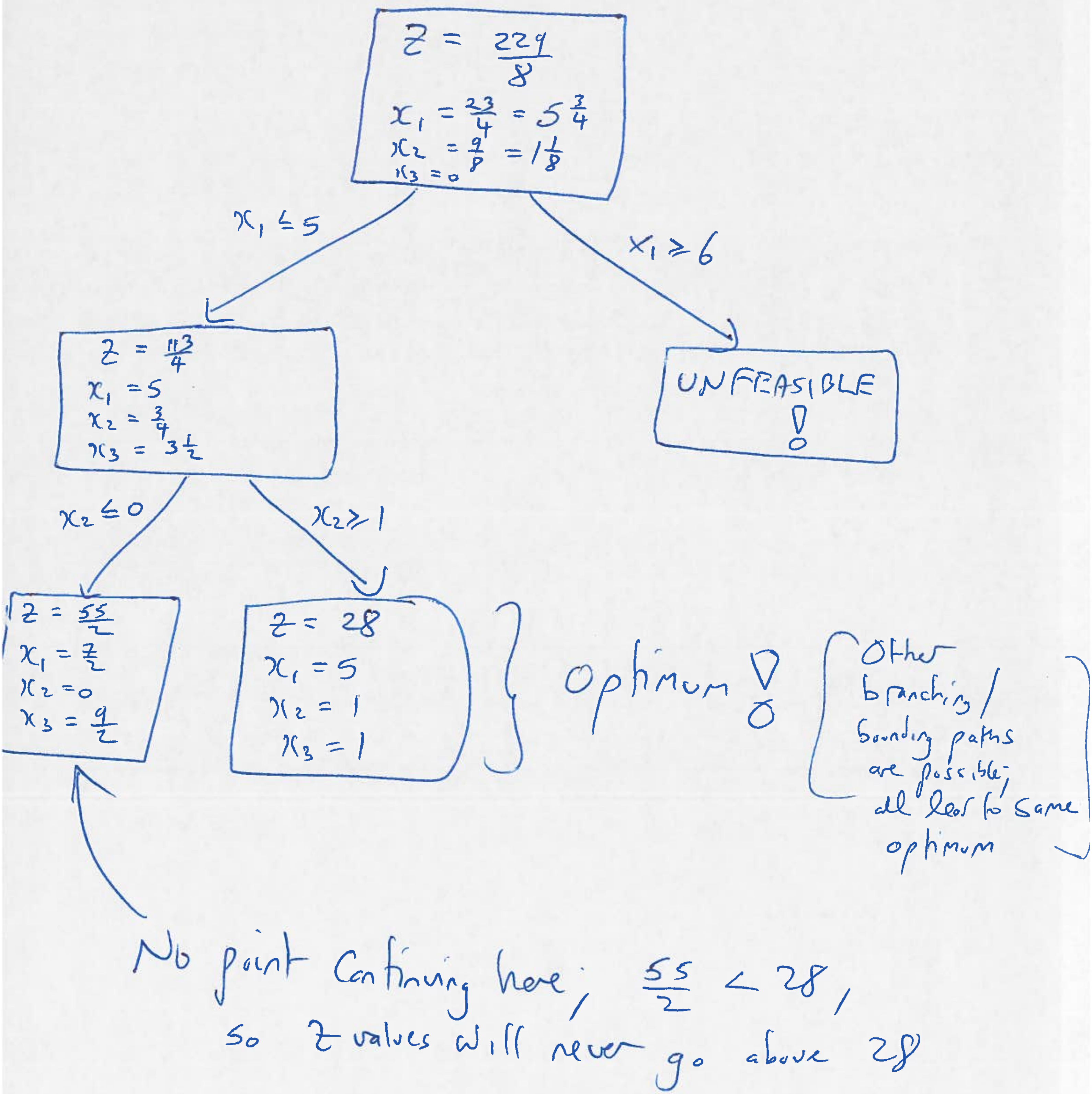
Adding ⓐ leads to the addition of two possible new cutting planes, $\frac{1}{3}x_3 - x_7 = \frac{2}{3}$ and $\frac{2}{3}x_3 - x_7 = \frac{1}{3}$

Adding the first of these leads to infeasibility; adding the second leads to the addition of $\frac{x_2}{2} - x_8 = \frac{1}{2}$, which leads to infeasibility

Adding ⓑ to ②, immediately leads to infeasibility.

Conclusion: all paths lead eventually to a problem that, when integer constraints are ignored, has no feasible points. This supports possibility 2 from last question.

5. Problem set 6.4, question 2, part a) (page 244).



6. Problem set 6.4, question 2 part c) (page 244).

Initial optimum, ignoring integer constraints:

$z = 132\frac{2}{3}$, $x_1 = 3\frac{5}{6}$

$x_2 = 0$

$x_3 = \frac{5}{6}$

One choice for branching: $x_1 \le 3$ vs $x_1 \ge 4$

(Note: x_3 not required to be integer!)

This (eventually) leads to integer optimum:

$z = 132$ at $x_1 = 4$

$x_2 = 0$

$x_3 = \frac{1}{3}$