MATH 30210, FALL 2014, HOMEWORK 9 SOLUTIONS

1)

					u's
[5] 9	[6] 3	[7]	[5]	12	0
[9]	[10] 5	[9] 9	[6] x(-1)	14	4
[3] x(-1)	[4] x(-1)	[4] 2	[2] 8	10	-1
9	8	11	8		
v's: 5	6	5	3		

Cost: 218

Above is NW rule basic feasible solution, and assignment of variables $u_1, u_2, u_3, v_1, v_2, v_3, v_4$ satisfying $u_i + v_j = c_{ij}$ for each basic cell (i,j). Cells with x's have $u_i + v_j > c_{ij}$. All three have $u_i + v_j$ exceeding c_{ij} by the same amount, so I choose (arbitrarily) cell $(2,4)$ to be entering variable.

Loop: $(2,4) \to (3,4) \to (3,3) \to (2,3) \to (2,4)$

These are the cells that lose flow if flow enters $(2,4)$; so up to 8 units can be put through $(2,4)$ without losing feasibility.

New basic feasible solution:

				u's
$\boxed{5}$ 9	$\boxed{6}$ 3	$\boxed{7}$	$\boxed{5}$	0
$\boxed{9}$	$\boxed{10}$ 5	$\boxed{9}$ 1	$\boxed{6}$ 8	4
$\boxed{3}$ x	$\boxed{4}$ x	$\boxed{4}$ 10	$\boxed{2}$	-1
v's 5	6	5	2	

Cost 210

Not yet optimal. Enter (3, 2) [arbitrary choice]
use loop $(3,2) \to (3,3) \to (2,3) \to (2,2) \to (3,2)$,
depart (2, 2), get:

$\boxed{5}$ 9	$\boxed{6}$ 3	$\boxed{7}$	$\boxed{5}$	0
$\boxed{9}$	$\boxed{10}$	$\boxed{9}$ 6	$\boxed{6}$ 8	3
$\boxed{3}$	$\boxed{4}$ 5	$\boxed{4}$ 5	$\boxed{2}$	-2
5	6	6	3	

In all cells, have $u_i + v_j \le c_{ij}$, so this is an optimal scheme

Cost: $9 \times 5 + 6 \times 3 + 9 \times 6 + 5 \times 4 + 4 \times 5 + 6 \times 8$
$= 205$.

2) Supply exceeds demand by 10, so add dummy outlet with demand 10, all costs 0.

Some routes have cost ∞. Replace these with a cost larger than the total cost of any feasible scheme ; 1600 > 75×16 will do

							u
[5] 10	[−2] 9	[−7] −x 1	[6] +x	[4]	[0]	20	0
[14]	[1]	[−6] 10	[16]	[1600]	[0]	10	1
[1600]	[−1]	[0] 4 +x	[1600] 20 −x	[13] 1	[0]	25	7
[12]	[−2]	[−9]	[14]	[12] 10	[0] 10	20	6
10	9	15	20	11	10		
v: 5	−2	−7	1593	6	−6		

Enter cell (1, 4) at value 1
Depart cell (1, 3)

						u
$\boxed{5}$ 10	$\boxed{-2}$ ~~9~~	$\boxed{-7}$	$\boxed{6}$ ~~1~~ 10	$\boxed{4}$	$\boxed{0}$	0
$\boxed{14}$	$\boxed{1}$	$\boxed{-6}$ 10	$\boxed{16}$	$\boxed{1600}$	$\boxed{0}$	1588
$\boxed{1600}$	$\boxed{-1}$ 9	$\boxed{0}$ 5	$\boxed{1600}$ ~~19~~ 10	$\boxed{13}$ 1	$\boxed{0}$	1594
$\boxed{12}$	$\boxed{-2}$	$\boxed{-9}$	$\boxed{14}$	$\boxed{12}$ 10	$\boxed{0}$ 10	1593
v: 5	-2	-1594	6	-1581	-1593	

Enter (3, 2) at value 9

Depart (1, 2) [redacted]

						u
$\boxed{5}$ 10			$\boxed{6}$ 10			0
		$\boxed{-6}$ 10				1600
	$\boxed{-1}$ 9	$\boxed{0}$ 5	$\boxed{1600}$ 10	$\boxed{13}$ 1		1594
				$\boxed{12}$ 10	$\boxed{0}$ 10	1569
v: 5	-1595	-1594	6	-1581	-1569	

Enter (4, 4) at value 10

Depart (3, 4)

Note: I'm not using rule of thumb "enter cell with $u_i + v_j - c_{ij}$ as big as possible", I'm choosing (4, 4) to quickly get rid of flow in (3, 4).

Get the following feasible scheme, which does not use any of the routes that originally cost ∞:

						u's
$\boxed{5}$ 10	$\boxed{-2}$ ✓	$\boxed{[illegible]}$ ✓	$\boxed{6}$ 10	$\boxed{4}$ ✓	$\boxed{0}$ ✓	0
$\boxed{14}$ ✓	$\boxed{1}$	$\boxed{-6}$ 10	$\boxed{16}$	$\boxed{1600}$	$\boxed{0}$ ✓	3
$\boxed{1600}$ ✓	$\boxed{-1}$ 9	$\boxed{0}$ 5	$\boxed{1600}$ ✓	$\boxed{13}$ 11	$\boxed{0}$ X	9
$\boxed{12}$ X	$\boxed{-2}$	$\boxed{-9}$	$\boxed{14}$ 10	$\boxed{12}$ 0	$\boxed{0}$ 10	8
v's: 5	−10	−9	6	4	−8	

Optimality criterion not yet satisfied

Note that I'm choosing to keep this at 0, to keep the right number of basic variables
[In case I wanted to continue to find optimal solution]

3) a) Introduce phantom 5^{th} outlet, with demand 5.
Make shipping cost from A to 5 be $\underline{\underline{c}}$
Make all other shipping costs to 5 be $\underline{\underline{0}}$

b) As above, but replace c with $\underline{\underline{-I}}$

c) As above, but replace c with ∞
[Practically: replace c with a number so large that if ever a single unit is shipped at A (ie, send to 5), the total cost exceeds the maximum possible cost of a feasible scheme that does not use the link A5. One way to do this is to take the largest ~~actual~~ actual cost, and multiply it by the total supply, then add $\underline{1}$.

Any optimal solution to the modified problem that does not send anything from A to 5 gives an optimal shipping scheme with no surplus at A.
Conversely, if the optimum found by Transportation algorithm uses A5, then we know that no feasible scheme exists that avoids A5]

4) Turn this into a transportation problem by thinking of each

$\left\{ \begin{array}{c} \text{~~[struck word]~~ lawyer} \\ \text{case} \end{array} \right\}$ as an $\left\{ \begin{array}{c} \text{warehouse} \\ \text{outlet} \end{array} \right\}$ with $\left\{ \begin{array}{c} \text{supply} \\ \text{demand} \end{array} \right\}$ 1

and using the lawyer-hours as costs:

					u
1	14 1	12	13	9	0
1	8 0	6 1	8	4	−6
1	12	10 0	9 1	6	−2
1	11	8 X	11 0	8 1	0
	1	1	1	1	
v	14	12	11	8	

NW rule initial basic feasible assignment

Cell (4,3) is one in which $u_i + v_j$ exceeds c_{ij} by the most, so enter this at value 0, depart (3,2) (arbitrary choice)

				u
14 1	12	13	9 X	0
8 0	6 1	8	4	−6
12	10	9 1	6	−6
11	8 0	11 0	8 1	−4
v 14	12	15	12	

Cell (1,4) enters,

loop is

(1,4) → (4,4) → (4,2) → (2,2) → (2,1) → (1,1) → (1,4)

Cell (1,4) can enter at value ~~7~~

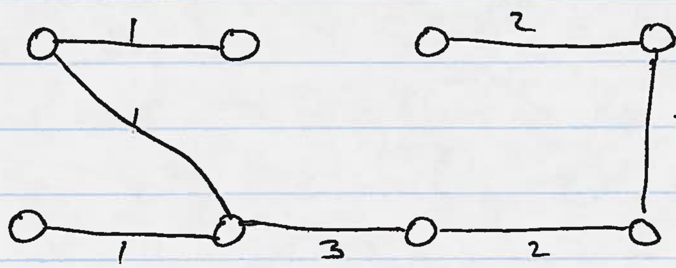
Get feasible solution:

0			1
1	0		
		1	
	1	0	

← departing cell (arbitrarily chosen)

[Meaning: lawyer 1 → Case 4
" 2 → " 1
3 → " 3
4 → " 2]

5) One possible solution:



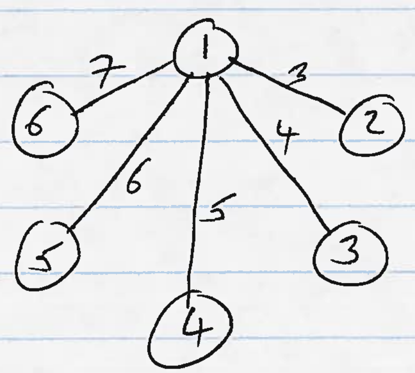
Total cost: $1+1+1+2+2+2+3 = 12$

6) part i)

	1	2	3	4	5	6
1	–	~~3~~	~~4~~	~~5~~	~~6~~	~~7~~
2		–	~~5~~	~~6~~	~~7~~	~~8~~
3			–	~~7~~	~~8~~	~~9~~
4				–	~~9~~	~~10~~
5					–	~~11~~
6						–

Cost Matrix

The <u>unique</u> solution given by Kruskal's algorithm is:



Total cost <u>25</u>

Part ii) Seems like a good conjecture would be: Minimum cost connection network is obtained by connecting city 1 to each of the other cities directly, for a cost of

$$3+4+\dots+(n+1) = \frac{n^2+3n-4}{2}$$

This checks out for $n=3,4,5$

7) One possible solution: Create a cost matrix with

$$c_{ij} = \begin{cases} 0 & \text{if } i,j \text{ \underline{directly} connected} \\ 1 & \text{otherwise.} \end{cases}$$

Run Kruskal. If min cost connected network has total cost 0, then network is connected [all links in min cost connected network uses only pairs of citys that are directly connected]

If min cost connected network has cost ≥ 1, then it is necessary to use a pair of cities that are not directly connected, so original network was <u>not</u> fully connected.