Prob & Stats – SPRING 2008

Practice II
Solutions

Problem 1.

Let X and Y be continuous independent random variables, Y is uniformly distributed in $[0,1]$ and X has an exponential distribution with a parameter $\lambda = 1$.

Find $P(X + Y \geq 1)$.

Solution: X has density $f_X(x) = e^{-x}$ for $x \geq 0$, and Y has density $f_Y(y) = 1$ for $0 \leq y \leq 1$, so the joint density of X and Y is $f(x,y) = e^{-x}$ for $0 \leq x < \infty$ and $0 \leq y \leq 1$, and 0 otherwise.

$$P(X + Y \geq 1) = \int_0^1 \int_{1-y}^{\infty} e^{-x} dxdy = 1 - \frac{1}{e}.$$

Problem 2.

One-tenth of 1% of a certain type of RAM chip are defective. A student needs 50 chips for a certain board.

a) What is the probability that if she buys 53 chips, she'll have enough working chips for the board.

Solution: Let X be number of failed chips. X is binomial with $n = 53$, $p = .001$. Having at least 50 working chips is the same as having 3 or fewer failed chips.

$$P(X \leq 3) = \sum_{i=0}^{3} \binom{53}{i} (.001)^i (.999)^{53-i} \approx .999999718\ldots$$

b) What is the smallest number of chips that she should buy in order for there to be at least a 99% chance of having at least 50 working chips?

Solution: If N chips are bought, probability of having at least 50 working is

$$\sum_{i=50}^{N} \binom{N}{i} (.999)^i (.001)^{N-i}.$$

For $N = 50$ this is around 95%. For $N = 51$ it is already at least 99%. (Computations done using the binomial calculator at http://stattrek.com/Tables/Binomial.aspx).

Problem 3.

It has been determined that when one attempts to shoot an aerial target with a missile, the east-west miss distance, the north-south miss distance and the vertical miss distance are independent normal variables with the zero mean the standard deviation of .5 meters.

You are told that with probability .99 the missile will explode within x meters from the target. Find x.

Solution: Let D be distance to target that missile explodes. $D^2 = X_1^2 + X_2^2 + X_3^2$ where $X_i = \mathcal{N}(0, .25)$ for each i, and the X_i's are independent. $2X_i$ is a standard normal, so $4D^2 = (2X_1)^2 + (2X_2)^2 + (2X_3)^2$ is the sum of the squares of three independent standard normals, so is χ_3^2 (a chi-squared random variable with 3 degrees of freedom). From a χ_3^2 table we find that $P(\chi_3^2 \le 11.345) = .99$, so $P(D \le 1.68) = .99$. So $x \approx 1.68$.

Problem 4.
A soft drink machine can be regulated so that it discharges an average of μ ounces per cup. If the ounces of fill are described by a normal random variable with standard deviation $\sigma = 0.3$ ounce, find a value of μ such that 8-ounce cups will overflow only 1% of the time.

Solution: Let X be the ounces of fill; $X \sim \mathcal{N}(\mu, .09)$. A cup overflows if $X > 8$. So we want μ such that $P(X > 8) = .01$. $X > 8$ is the same as $(X - \mu)/.3 > (8 - \mu)/.3$, which is the same as $Z > (8 - \mu)/.3$ where Z is a standard normal. From a standard normal table, we see that $P(Z > 2.33) = .01$. So we need μ to satisfy $(8 - \mu)/.3 = 2.33$, or $\mu = 7.301$.

Problem 5.

The distribution of resistance for resistors of a certain type is known to be normal. 9.85 % of all resistors have a resistance exceeding 10.257 Ohms, and 5.05 % have resistance smaller than 9.671 Ohms. What are the mean value and variation of the resistance distribution?

Solution: Let R be resistance, with mean μ, variance σ^2. We know that

$$P(R > 10.257) = .0985 \quad \text{and} \quad P(R < 9.671) = .0505,$$

so

$$P\left(\frac{R-\mu}{\sigma} > \frac{10.257-\mu}{\sigma}\right) = .0985 \quad \text{and} \quad P\left(\frac{R-\mu}{\sigma} < \frac{9.671-\mu}{\sigma}\right) = .0505.$$

Since $(R-\mu)/\sigma$ is a standard normal, we get (consulting a standard normal table, and finding that $\Phi(1.29) = .9015 = 1 - .0985$ so $P(Z > 1.29) = .0985$, and that $\Phi(1.64) = .9495 = 1 - .0505$, so $P(Z < -1.64) = 1 - P(Z < 1.64) = .0505$)

$$\frac{10.257-\mu}{\sigma} = 1.29 \quad \text{and} \quad \frac{9.671-\mu}{\sigma} = -1.64$$

Solving these two equations for μ and σ, get $\mu = 10$ and $\sigma = .2$, so $\sigma^2 = .04$.

Problem 6.

Let X be a Poisson random variable with a parameter $\lambda = 2$. Find the expected value and the variance of $Y = 3^X$.

Solution:

$$E(Y) = E(3^X) = \sum_{k\ge 0} 3^k \frac{2^k}{k!} e^{-2} = e^{-2} \sum_{k\ge 0} \frac{6^k}{k!} = e^{-2} e^6 = e^4$$

$$E(Y^2) = E(9^X) = \sum_{k\geq 0} 9^k \frac{2^k}{k!} e^{-2} = e^{-2} \sum_{k\geq 0} \frac{18^k}{k!} = e^{-2}e^{18} = e^{16}$$

so $Var(Y) = e^{16} - e^8$.

Problem 7.

Assume that the time of arrival of birds at a particular place on a migratory route, as measured in days from the first of the year (January 1 is the first day), is approximated as a gaussian random variable X with $m = 200$ and $\sigma = 20$ days.

a) What is the probability the birds will arrive after 200th day?

Solution: Let X be the arrival day. $X \sim \mathcal{N}(200, 400)$.

$$P(X > 200) = \frac{1}{2}$$

(by symmetry ... a normal random variable is as likely to exceed its mean as to fall short of it).

b) What is the probability the birds arrive after 160 days but on or before 210th day.

Solution:

$$P(160 < X \leq 210) = P(-2 \leq \frac{X - 200}{20} \leq \frac{1}{2}) = \Phi(1/2) - \Phi(-2) = \Phi(1/2) - (1 - \Phi(2)) = .6688.$$

Problem 8.

If X is a random variable uniformly distributed over the interval $(0, 2)$, find $E(X^3 - X)$.

Solution:

$$E(X^3 - X) = \int_0^2 (x^3 - x)\frac{1}{2}dx = 1$$

Problem 9.

The number of cars which stop at a gas station is modeled by a Poisson random variable with an average of 10 cars per hour. What is the probability that at most one car will stop at this gas station during the next 30 minutes?

Solution: 10 cars per hour means the average number of cars over 30 minutes is 5; so we model the number of cars that arrive in a half-hour period by a Poisson random variable with parameter $\lambda = 5$. We want to know the probability that $X \leq 1$.

$$P(X \leq 1) = \frac{5^0}{0!}e^{-5} + \frac{5^1}{1!}e^{-5} = 6e^{-5}.$$

Problem 10.

If X has an exponential density and $E(X) = 20$, find $P(X > 20)$.

Solution: X has parameter $\lambda = 1/20$ ($E(X) = 1/\lambda$), so

$$P(X > 20) = \int_{20}^{\infty} \frac{1}{20} e^{-x/20} dx = e^{-1}.$$

Problem 11.

One microgram of radium contains 10^{16} atoms. The probability that a single atom will disintegrate during a one-millisecond time is $p = 10^{-15}$. Approximate the probability that more than two atoms will disintegrate in one millisecond.

Solution: This is an instance of many (10^{16}) independent repetitions of a trial (the disintegration of an atom) with a very low (10^{-15}) success probability, so we should model the number of atoms which disintegrate in a millisecond by a Poisson random variable X with parameter $10^{16} \times 10^{-15} = 10$.

$$P(X > 2) = 1 - P(X \leq 2) = 1 - \frac{10^0}{0!}e^{-10} - \frac{10^1}{1!}e^{-10} - \frac{10^2}{2!}e^{-10} = 1 - 61e^{-10}.$$

Problem 12.

If X is an exponential random variable with a parameter $\lambda = 2$, find the CDF $F_Y(y)$ of $Y = 2X - 1$.

Solution: X only takes on positive values, so Y only takes on values that are greater than or equal to -1. This gives

$$F_Y(y) = 0 \quad \text{if } y < -1.$$

For $y \geq -1$,

$$P(Y \leq y) = P(X \leq \frac{1}{2}(y+1)) = \int_0^{\frac{1}{2}(y+1)} 2e^{-2x}dx = 1 - e^{-(y+1)}$$

so

$$F_Y(y) = 1 - e^{-(y+1)} \quad \text{if } y \geq -1.$$