**Math 30-440: Probability and Statistics**
**Spring Semester 2008**
**Solutions to Homework 6**

1. **Problem 23b) (Ch 5)**: Here and everywhere I'll use $Z$ for the standard normal.

$$\begin{aligned} P(4 < X < 16) &= P\left(\frac{4-10}{6} < \frac{X-10}{6} < \frac{16-10}{6}\right) \\ &= P\left(-1 < Z < 1\right) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = .6826. \end{aligned}$$

2. **Problem 23c) (Ch 5)**:

$$\begin{aligned} P(X < 8) &= P\left(\frac{X-10}{6} < \frac{8-10}{6}\right) \\ &= P\left(Z < 1/3\right) \\ &= \Phi(-1/3) = 1 - \Phi(1/3) = .3707. \end{aligned}$$

**Note**: There may be some round-off error here and in other questions. To get the numerics right, we really should use a computer.

3. **Problem 24 (Ch 5)**: Let $X_i$ be score of student $i$.

$$\begin{aligned} P(X_i < 600) &= P\left(\frac{X_i-500}{100} < \frac{600-500}{100}\right) \\ &= P\left(Z < 1\right) \\ &= \Phi(1) = .8413. \end{aligned}$$

If the students are chosen independently, then the probability that all five of them score below 600 is $(.8413)^5 = .42$.

$$\begin{aligned} P(X_i > 640) &= P\left(\frac{X_i-500}{100} > \frac{640-500}{100}\right) \\ &= P\left(Z > 1.4\right) \\ &= 1 - \Phi(1.4) = .0808. \end{aligned}$$

If the students are chosen independently, then the number that score above 640 is a binomial random variable with $n = 5$, $p = .0808$, and so the probability that exactly 3 score above 640 is

$$\binom{5}{3}(.0808)^3(1 - .0808)^{5-3} = .004.$$

4. **Problem 27 (Ch 5)**: Let $X$ be life of bulb.

$$\begin{aligned} P(X > L) &= P\left(\frac{X-2000}{85} > \frac{L-2000}{85}\right) \\ &= P\left(Z > \frac{L-2000}{85}\right). \end{aligned}$$

What $z$ satisfies $P(Z > z) = .95$? $\Phi(1.64) = .95$, so $P(Z < 1.64) = .95$, so $P(Z > -1.64) = .95$. So we should choose $L$ so

$$\frac{L-2000}{85} = -1.64,$$

or $L = 1860$.

5. **Problem 31 (Ch 5)**: Let $X$ be life of randomly chosen chip.

$$\begin{aligned} P(X > 4 \times 10^6) &= P\left(\frac{X - 4.4 \times 10^6}{3 \times 10^5} > \frac{4 \times 10^6 - 4.4 \times 10^6}{3 \times 10^5}\right) \\ &= P\left(Z > -4/3\right) \\ &= \Phi(4/3) = .9082. \end{aligned}$$

So in a large batch, expect 90.82% to be good; contract should be made.

6. **Problem 34a) (Ch 5)**: Let $X_i$ be annual rainfall in year $i$, $i = 1, 2, 3$. We will assume that the $X_i$'s are independent.

$$\begin{aligned} P(X_1 > 42) &= P\left(\frac{X_1 - 40.14}{8.7} > \frac{42 - 40.14}{8.7}\right) \\ &= P\left(Z > .214\right) \\ &= 1 - \Phi(.214) = .4178. \end{aligned}$$

7. **Problem 34b) (Ch 5)**: $X_1$ and $X_2$ are independent (assumption) and have mean 40.14, variance $8.7^2 = 75.69$, so $X_1 + X_2$ has mean 80.28 and variance 151.38, so standard deviation 12.3.

$$\begin{aligned} P(X_1 + X_2 > 84) &= P\left(\frac{X_1 + X_2 - 80.28}{12.3} > \frac{84 - 80.28}{12.3}\right) \\ &= P\left(Z > .34\right) \\ &= 1 - \Phi(.34) = .3669. \end{aligned}$$

8. **Problem 34c) (Ch 5)**: $X_1$, $X_2$ and $X_3$ are independent (assumption) and have mean 40.14, variance $8.7^2 = 75.69$, so $X_1 + X_2 + X_3$ has mean 120.42 and variance 227.07, so standard deviation 15.07.

$$\begin{aligned} P(X_1 + X_2 + X_3 > 126) &= P\left(\frac{X_1 + X_2 + X_3 - 120.42}{15.07} > \frac{126 - 120.42}{15.07}\right) \\ &= P(Z > .42) \\ &= 1 - \Phi(.42) = .3372. \end{aligned}$$

9. **Problem 37a) (Ch 5)**: Let $X$ be number of hours. The density of $X$ is $f(x) = e^{-x}$ $(x \geq 0)$

$$P(X > 2) = \int_2^\infty e^{-x} dx = [-e^{-x}]_2^\infty = 1/e^2.$$

10. **Problem 37b) (Ch 5)**: For this we use the memoryless property.

$$P(X > 3|X > 2) = P(X > 1) = \int_1^\infty e^{-x} dx = [-e^{-x}]_1^\infty = 1/e.$$

11. **Problem 39 (Ch 5)**: Let $X$ be life of car. If $X$ is exponential, then (by memorylessness)

$$P(X > 30|X > 10) = P(X > 20) = \int_{20}^\infty \frac{1}{20} e^{-x/20} dx = [-e^{-x/20}]_{20}^\infty = 1/e.$$

If $X$ is uniform on $(0, 40)$, then

$$P(X > 30|X > 10) = \frac{P(X > 30, X > 10)}{P(X > 10)} = \frac{P(X > 30)}{P(X > 10)} = \frac{.25}{.75} = 1/3.$$

12. **Problem 44 (Ch 5)**: $X + Y$ is chi-squared with $3 + 6 = 9$ degrees of freedom. From an online calculator,

$$P(\chi_9^2 > 10) = .3505.$$

**13. Problem 1 (Ch 6)**: $E(X_i) = 0 \times .2 + 1 \times .3 + 3 \times .5 = 1.8$. So $E(\bar{X}) = 1.8$ whether $n = 2$ or 3 (the expectation of the sample mean is always equal to the expectation of the underlying distribution).

$E(X^2) = 0 \times .2 + 1 \times .3 + 9 \times .5 = 4.8$ so $Var(X_i) = 4.8 - (1.8)^2 = 1.56$. Since the variance of the sample mean is always the variance of the underlying distribution divided by the size of the sample, $Var(\bar{X}) = .78$ when $n = 2$ and $.52$ when $n = 3$. (Here I'm using the formulas derived on page 203).

**14. Problem 4a) (Ch 6)**: The random variable associated with a single roll, $X_1$, takes on value 35 with probability 1/38 and value $-1$ with probability 37/38. So

$$E(X_1) = 35\frac{1}{38} - 1\frac{37}{38} = -1/19,$$

and $E(X^2) = (35)^2\frac{1}{38} + (-1)^2\frac{37}{38} = \frac{631}{19}$ so

$$Var(X_1) = \frac{631}{19} - \left(\frac{-1}{19}\right)^2 = \frac{11988}{361}$$

By the central limit theorem, if the wheel is spun $n$ times then the distribution of the total winnings, $X = X_1 + \ldots + X_n$, is approximately normal with mean $-n/19$ and variance $11988n/361$. The event of "winning" is the event $X > 0$.

$$\begin{aligned} P(X > 0) &= P\left(\frac{X + n/19}{\sqrt{11988n/361}} > \frac{0 + n/19}{\sqrt{11988n/361}}\right) \\ &= P\left(Z > \sqrt{\frac{n}{11988}}\right) = 1 - \Phi\left(\sqrt{\frac{n}{11988}}\right). \end{aligned}$$

When $n = 34$, this is roughly $1 - \Phi(.05) = .48$.

**15. Problem 4a) (Ch 6)**: When $n = 1000$, the probability is roughly $1 - \Phi(.29) = .386$.

**16. Problem 4a) (Ch 6)**: When $n = 100000$, the probability is roughly $1 - \Phi(2.89) = .002$.

17. **Problem 6 (Ch 6)**: Let $X_i$ be the $i$th roundoff error. The sum of the errors is $X = X_1 + X_2 + \ldots + X_{50}$. We want $P(-3 < X < 3)$ (i.e., the probability that the accumulated error is no more than $\pm 3$). Since each $X$ has mean 0 and variance 1/12 (property of uniform random variables), by the central limit theorem $X$ is approximately normal with mean 0 and variance $50/12 = 4.16\ldots$, so

$$\begin{aligned} P(-3 < X < 3) &= P\left(-3/\sqrt{4.16\ldots} < \frac{X}{\sqrt{4.16\ldots}} < 3/\sqrt{4.16\ldots}\right) \\ &= P(-1.47 < Z < 1.47) = 2\Phi(1.47) - 1 = .8584 \end{aligned}$$

so the probability of roundoff error greater than 3 is around .141.

18. **Problem 7 (Ch 6)**: Let $X_i$ be the $i$th roll of the dice. $E(X_i) = 3.5$ and $Var(X_i) = 35/12$. If we roll 140 times, the sum of the dice is $X = X_1 + \ldots + X_{140}$ which by central limit theorem is approximately normal with mean 490 and variance $408.3\ldots$. The probability that we have not yet reached 400 is

$$\begin{aligned} P(X < 400) &= P\left(\frac{X - 490}{\sqrt{408.3\ldots}} < \frac{-90}{\sqrt{408.3\ldots}}\right) \\ &= P(Z < -4.45) = 1 - \Phi(4.45) \approx 0. \end{aligned}$$

So the probability that we require more than 140 rolls is very close to 0.