Math 30-440: Probability and Statistics
Spring Semester 2008
Exam 1 — Solutions

1. (14 pts) A standard deck of cards contains 52 cards (4 suits, 13 denominations in each suit). A poker hand contains 5 cards. Assuming that all poker hands are equally likely, what is a probability of being dealt a hand with cards of *exactly* two denominations?

Solution: The number of two-denomination hands which split 4-1 (i.e., four-of-a-kind) is 13.12.4 (13 choices for which denomination there are four of, 12 choices for which denomination there is one of, 4 choices for that one card.) The number of two-denomination hands which split 3-2 (i.e., full house) is 13.12.4.6 (13 choices for which denomination there are three of, 12 choices for which denomination there are two of, $4 = \binom{4}{3}$ choices for the three cards and $6 = \binom{4}{2}$ choices for the two cards.) So the probability of a two-denomination hand is

$$\frac{13.12.4 + 13.12.4.6}{\binom{52}{5}} \approx .168\%.$$

2. (16 pts) It has been observed that on a Probability test,

- 80 % of students first attempt a problem that they are 90 % likely to get right
- 15 % first try a problem that they are 50 % likely to get right, and
- 5 % of students first try a problem that they have 10 % chance to solve.

a) What is the probability that the first problem attempted by a randomly selected student will *not* be solved correctly?

Solution: Write A for the event that a student is 90% likely to get the first problem, and B for 50% and C for 10%. Write G for event that the first problem is solved correctly (good), and W for the event that it is solved incorrectly.

$$P(W) = P(W|A)P(A) + P(W|B)P(B) + P(W|C)P(C) = (.1)(.8) + (.5)(.15) + (.9)(.05) = .2$$

b) A student solves the first problem she attempts correctly. How likely it is that she had a 50-50 chance of solving it?

Solution: This is an application of Bayes' formula, though the computation of the denominator has already been done in part a).

$$P(B|G) = \frac{P(BG)}{P(G)} = \frac{P(G|B)P(B)}{1 - P(W)} = \frac{(.5)(.15)}{.8} = .09375.$$

3. (20 pts) Decide whether each of the following statements is true or false, and give a *short* explanation of why:

1. ____ If events E and G are independent, then knowing $P(E)$ and $P(G)$ allows one to compute $P(E \cup G)$.

   TRUE: $P(E \cup G) = P(E) + P(G) - P(EG) = P(E) + P(G) - P(E)P(G)$ (the last part using independence).

2. ____ Mutually exclusive events are independent.

   FALSE: If E, F are mutually exclusive, then $EF = \emptyset$ so $P(E)P(F) = 0 \neq P(E)P(F)$ (except in one special case, when one of $E, F = \emptyset$).

3. ____ A function $f(x) = \begin{cases} c(1-x^2) & \text{if } |x| \leq 2 \\ 0 & \text{otherwise} \end{cases}$ could be a valid probability density function.

   FALSE: If $c > 0$, f is negative near 2. If $c < 0$, f is negative near 0. Density functions cannot ever be negative, so we must have $c = 0$. But then $\int_{-2}^{2} f(x)dx = 0$, and not 1 as it should be. So no value of c will lead to a valid density function.

4. ____ If random variables X and Y are independent, then $E(XY) = E(X)E(Y)$.

   TRUE: e.g., if X, Y are continuous with joint density function $f(x, y)$, and the density of X and Y are $f_X(x)$ and $f_Y(y)$ respectively, then by independence $f(x, y) = f_X(x)f_Y(y)$ and

$$E(XY) = \int_{-\infty}^{\infty} xyf(x, y)dxdy = \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy = E(X)E(Y).$$

4. (16 pts) It is always warm in tropical paradise. However, due to some random factors, the daytime temperature does fluctuate every day with the same mean of 85° F and the same variance of 16. You do not pay much attention to weather reports, but want to tell your friends that during your stay the average daytime temperature was between 83° F and 87° F. How long should your vacation be for you to be 95 % certain that you're telling them the truth?

Solution: Let X_i be the daytime temperature on day i. The X_i's are independent (we assume) random variables each with mean 85 and variance 16. If your stay is n days, the average daytime temperature is $\frac{X_1+\ldots+X_n}{n}$. We want to know what value of n will ensure

$$P\left(\left|\frac{X_1 + \ldots + X_n}{n} - 85\right| \leq 2\right) \geq .95 \quad \text{or, equivalently} \quad P\left(\left|\frac{X_1 + \ldots + X_n}{n} - 85\right| > 2\right) \leq .05.$$

The weak law of large numbers says

$$P\left(\left|\frac{X_1 + \ldots + X_n}{n} - 85\right| > 2\right) \leq \frac{16}{n(2)^2} = \frac{4}{n}.$$

To make $4/n \leq .05$, we should take $n = 80$ (a long vacation!).

5. (14 pts) The moment generating function of a random variable X is given by

$$\phi(t) = M_X(t) = \frac{a}{2-t^2} \ .$$

Find a, $E(X)$ and $Var(X)$.

Solution: Since $\phi(t) = E(e^{tX})$, have $\phi(0) = E(e^{0X}) = E(e^0) = E(1) = 1$. But $\phi(0) = a/2$. So $a = 2$. For the other two parts:

$$E(X) = \phi'(0) = \left[\frac{2at}{(2-t^2)^2}\right]_{t=0} = 0$$

and

$$E(X^2) = \phi''(0) = \left[\frac{2a(2-t^2)^2 + 8at^2(2-t^2)}{(2-t^2)^4}\right]_{t=0} = \frac{8a}{16} = \frac{a}{2} = 1$$

so $Var(X) = E(X^2) - (E(X))^2 = 1$.

6. (20 pts) A joint probability density function of random variables X and Y is given by the formula

$$f(x,y) = \begin{cases} \frac{1}{4} & \text{if } |x| \le 2, \ 0 \le y \le 2-|x| \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch a graph showing the region of the plane for which $f(x,y)$ is non zero.

Solution: f is non-zero in the triangle with vertices $(-2,0), (2,0)$ and $(0,2)$.

b) Find $P(|X| \le Y)$

Solution: The region of the plane in which $|x| \le y$ and at the same time f is not zero is the interior of the square with vertices $(0,0), (1,1), (0,2)$ and $(-1,1)$. Call this square S.

$$P(|X| \le Y) = \int\int_S \frac{1}{4} dA = \frac{1}{4}(\text{Area of } S) = \frac{1}{4}(\sqrt{2})^2 = \frac{1}{2}.$$

d) Find the cumulative distribution function of the random variable $Z = X + Y$.

Solution: $X+Y$ can never be less than -2 and is always less than or equal to 2, so $F_Z(a)$ is 0 for $a < -2$ and 1 for $a \ge 2$. For the remaining a: $F_Z(a) = P(X+Y \le a)$. The region of the plane in which $X+Y \le a$ and f is non-zero is the triangle T with vertices $(-2,0), (a,0)$ and $((a-2)/2, (a+2)/2)$. So

$$F_Z(a) = \int\int_T \frac{1}{4} dA = \frac{1}{4}(\text{Area of } T) = \frac{1}{4}\left(\frac{1}{2}(a+2)\frac{(a+2)}{2}\right) = \frac{(a+2)^2}{16}.$$