Math 30-440: Probability and Statistics
Spring Semester 2008
Exam 2 — Solutions

1. (20 pts) **a)** A random sample of size 4 is taken from an unknown distribution, and the observed values are 2, -8, 0 and 6. Compute the sample mean and sample variance.

Solution: $\bar{X} = (X_1 + X_2 + X_3 + X_4)/4 = (2 - 8 + 0 + 6)/4 = 0$.

$$S^2 = \frac{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + (X_3 - \bar{X})^2 + (X_4 - \bar{X})^2}{3} = \frac{4 + 64 + 0 + 36}{3} = \frac{104}{3}.$$

b) If X and Y are standard normal random variables, what can you say about $X - Y$? State any additional assumptions you make.

Solution: $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$. If X and Y are independent, then $X - Y \sim \mathcal{N}(0 - 0, 1 + 1) = \mathcal{N}(0,2)$.

c) Indiana has a population of 6,000,000. 6,000 residents of the state are more than seven foot tall. A random sample of 865 Hoosiers is selected. Write down an expression for the probability that no more than 3 of those sampled are more than seven foot tall.

Solution: Let X be the number from our sample that are more than seven foot tall. If we model X as a binomial random variable with $n = 865$ and $p = 6000/6000000 = .001$ then

$$P(X \leq 3) = \sum_{i=0}^{3} \binom{865}{i} (.001)^i (1 - .001)^{865-i}.$$

If we model X as a Poisson random variable with $\lambda = 865 \times .001 = .865$ then

$$P(X \leq 3) = \sum_{i=0}^{3} \frac{.865^i}{1!} e^{-.865}.$$

d) The life of fan belt in a car engine (measured in miles) is distributed exponentially with parameter $\lambda = \frac{1}{30000}$. If a fan belt is observed to be in working order 25,000 miles after installation, how many more miles is it expected to work for?

Solution: By the memoryless property of the exponential random variable, the distribution of the subsequent life of the fan belt, given that it is working after 25,000 miles, is exactly the same as the original distribution: exponential with $\lambda = \frac{1}{30000}$. So the expectation for the remaining life is 30,000 miles (**NOT** 5,000!)

2. (20 pts) A local supermarket sells grapefruits in plastic bags (four to a bag). The weight of a single grapefruit is a normal random variable with mean 0.5 lb and standard deviation 0.15 lb. The weight that can be sustained by a bag is also a normal random variable, with mean 2.2 lb and standard deviation 0.4 lb.

You are on a grapefruit only diet and need to buy a bag of grapefruit. What is the probability that the bag will break when you lift it?

Solution: Let X_i be the weight of a single grapefruit ($i = 1, 2, 3, 4$); $X_i \sim \mathcal{N}(.5, (.15)^2)$. The weight of four (independently chosen) grapefruits is $X = X_1+X_2+X_3+X_4 \sim \mathcal{N}(4(.5), 4(.15)^2) = \mathcal{N}(2, .9)$. Let Y be the weight a bag can sustain; $Y \sim \mathcal{N}(2.2, (.4)^2)$. The bag breaks if $X > Y$, that is, if $X - Y > 0$. Note that the distribution of $X - Y$ is $\mathcal{N}(-.2, .9 + (.4)^2) = \mathcal{N}(-.2, .25)$.

$$P(X - Y > 0) = P\left(\frac{(X - Y) - (-.2)}{.5} > \frac{0 - (-.2)}{.5}\right) = P(Z > .4) = 1 - \Phi(.4) = .3446.$$

3. (20 pts) An introverted professor X rarely turns her face away from the blackboard. The moment when she first faces her students is equally likely to occur at any point during her hour-long lecture. X's student, Y, is very busy. He's always at least 10 minutes late, though he always manages to get to class (at a completely random moment) before the lecture is halfway through. How likely is it that when X faces the class for the first time, she'll see Y eagerly taking notes?

Solution: X is uniform on $(0, 60)$ and Y is uniform on $(10, 30)$, so the joint distribution of X and Y is uniform on the strip $(0, 60) \times (10, 30)$ and the joint density is $1/(20 \times 60)$ on this strip and 0 off it. We want $P(X > Y)$, which is the probability that a point in the plane chosen according to this joint density lies in this strip and below the line $x = y$. The area in question is a trapezoid bounded by the points $(10, 10)$, $(30, 30)$ $(60, 30)$ and $(60, 10)$, which has area 800; so the probability of landing in this area is $800/(20 \times 60) = 2/3$. (You should draw a picture to help understand this).

4. (20 pts) The average income of Notre Dame alumni (measured in thousands of dollars) is 80, with variance 120. Thirty Domers find themselves sitting randomly together on game day, and pass the time during a media time out by computing their average annual income.

How likely is it that they get an answer greater than 85?

Solution: Let $X_1, \ldots, X_{30}$ be the income of the 30 Domers. Each X_i has mean 80 and variance 120, and we assume that they are independent. So the average, or sample mean $\bar{X}$, has mean 80 and variance $120/30 = 4$. By the central limit theorem, $\bar{X} \approx \mathcal{N}(80, 4)$. We want $P(\bar{X} > 85)$.

$$P(\bar{X} > 85) = P\left(\frac{\bar{X} - 80}{2} > \frac{85 - 80}{2}\right) = P(Z > 2.5) = 1 - \Phi(2.5) = .0062.$$

5. (20 pts) Dr. G. teaches on MWF 3-3:50 pm. On Mondays he usually makes a mistake, on average, every 10 minutes. On Wednesdays he mostly gets his act together, but still adds things incorrectly once about every 20 minutes. By Friday he gets tired and, on average, misspells a word on a blackboard every 5 min. On each day, the number of mistakes Dr. G makes is modeled by a Poisson random variable.

If a randomly chosen lecture happened to be surprisingly error free, what is the probability that it was delivered on Friday?

Solution: Let M be the event that the Monday class is observed, etc. Each class is equally likely to have been chosen for scrutiny, so $P(M) = P(W) = P(F) = 1/3$. Let X be number of mistakes made. Since on Monday the average number is $50/10 = 5$, $P(\{X = 0\}|M) = e^{-5}$ (the number of mistakes made on Monday is Poisson with $\lambda = 5$). Similarly, $P(\{X = 0\}|W) = e^{-2.5}$ and $P(\{X = 0\}|F) = e^{-10}$. We uses Bayes' formula to conclude that the probability that the chosen class is on a Friday given no mistakes is

$$\begin{aligned} P(F|\{X = 0\}) &= \frac{P(\{X = 0\}|F)P(F)}{P(\{X = 0\}|M)P(M) + P(\{X = 0\}|W)P(W) + P(\{X = 0\}|F)P(F)} \\ &= \frac{e^{-10}}{e^{-5} + e^{-2.5} + e^{-10}}. \end{aligned}$$