Introduction to Probability and Statistics

Combinatorial arguments

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The combinatorial expression $\binom{n}{r}$ is *defined* by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

but it also has a *combinatorial interpretation*, namely

$$\binom{n}{r} \text{ counts the number of subsets of size } r \text{ of a set of size } n.$$

As a result, there are often two ways to verify an equality involving $\binom{n}{r}$: a direct verification, using the definition, and what is called a *combinatorial argument*, which is an argument that shows that both sides of the equality can be interpreted as *counting the same thing*.

Here's an example. Consider the equality

$$r\binom{n}{r} = n\binom{n-1}{r-1}.$$

This can be verified directly from the definition:

$$r\binom{n}{r} = r\frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!} = n\frac{(n-1)!}{(r-1)!(n-r)!} = n\binom{n-1}{r-1}.$$

But there is also a combinatorial argument that proves the identity. Imagine having to choose a committee of size r from a group n people, with the additional rule that one of the r people on the committee must be selected as committee chair. How many ways are there to select the committee-with-chair? One way to answer this question is to say that there are $\binom{n}{r}$ ways to choose the committee, and then, once it has been chosen, there are r ways to choose the chair. By the basic principle of counting, this means that

$$\text{Number of possible committees-with-chair} = r\binom{n}{r}.$$

But here's another way to answer the question. There has to be a chair, so we first choose who that is to be (n choices). The remaining $r-1$ members of the committee now have to be chosen from among the remaining $n-1$ people in the group ($\binom{n-1}{r-1}$ choices). It follows that

$$\text{Number of possible committees-with-chair} = n\binom{n-1}{r-1}.$$

Since both $r\binom{n}{r}$ and $n\binom{n-1}{r-1}$ count the same thing, they must be equal.

The second parts of Ross, Chapter 3, Problems 16 and 17, should be answered in a similar way to the argument above.