Math 30440 — Probability and Statistics

Spring 2010 first mid-term exam, February 16 2010

Instructors: David Galvin and Daniel Cibotaru

Name: SOLUTIONS

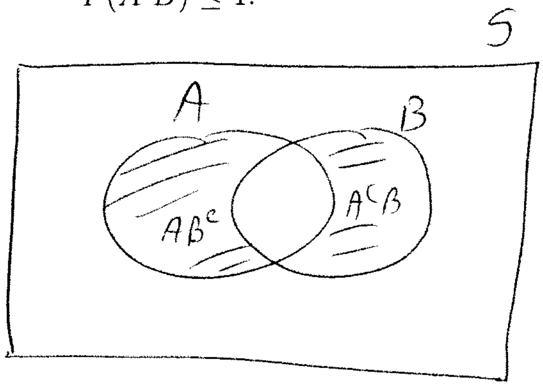
This examination contains 7 problems on 8 pages (including the front cover). It is closed-book. You may use up to 2 pages of handwritten notes. You may use a calculator, but only for arithmetic; you must calculate all integrals by hand. **Show all your work** on the paper provided. The honor code is in effect for this examination.

Scores

Question	Score	Out of
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

GOOD LUCK !!!

1. (a) Use a Venn diagram to show that for any two events A and B, $P(AB^c) + P(A^cB) \leq 1$.



AB^c and A^cB are mutually exclusive, so:

$$P(AB^c) + P(A^cB) = P(AB^c \cup A^cB)$$
$$\leq P(S) = 1$$

(b) Suppose that E and F are two events with $P(E) = .6$, $P(F) = .5$ and $P(EF) = .3$. What is the probability that exactly one of E, F occur?

From the picture in part a), we have

$P(\text{Exactly one of } E, F \text{ occur})$

$$= P(EF^c \cup E^cF)$$
$$= [P(EF^c) + P(EF)] + [P(E^cF) + P(EF)] - 2P(EF)$$
$$= P(E) + P(F) - 2P(EF)$$
$$= .6 + .5 - .3 \times 2 = .5$$

2. There is an 80% chance that the center of Hurricane D. will hit a certain coastal city. If it does then there is a 95% chance of massive rain in the city. If it doesn't there is still a 50% chances of massive rain.

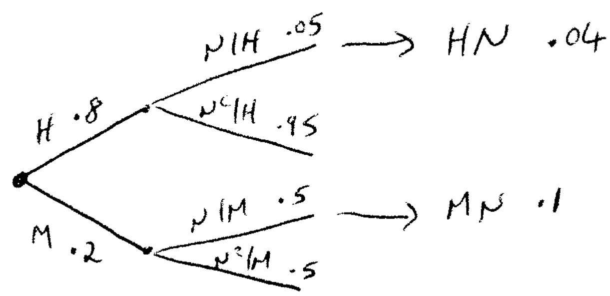
   (a) What is the probability that the city will not get massive rain?

$H = \{\text{Hits}\}$, $M = \{\text{Misses}\}$, $N = \{\text{No massive rain}\}$

$$P(N) = P(N|H)P(H) + P(N|M)P(M)$$

$$= .05 \times .8 + .5 \times .2$$

$$= .14$$



   (b) You hear on the Weather Channel that the city didn't get massive rain. What is the probability that the center of the hurricane struck the city?

$$P(H|N) = \frac{P(N|H)P(H)}{P(N)}$$

$$= \frac{.05 \times .8}{.14} = \frac{2}{7} = .285\ldots$$

3. From a group of 10 people (6 from Stanford and 4 from Keenan), I randomly select a team of 3 people to help paint the door of the Stanford-Keenan Chapel. Let X be the number people from Keenan on the team.

(a) Compute the mass function of X.

X takes on values $0, 1, 2, 3$

$$P(X=0) = \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} = \frac{20}{120} = \frac{1}{6}$$

$$P(X=1) = \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} = \frac{60}{120} = \frac{1}{2}$$

$$P(X=2) = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} = \frac{36}{120} = \frac{3}{10}$$

$$P(X=3) = \frac{\binom{4}{3}\binom{6}{0}}{\binom{10}{3}} = \frac{4}{120} = \frac{1}{30}$$

I'm picking 3 people, so P(3 from Keenan) is not $(.4)^3$ $\rightarrow$ that allows the possibility of picking the same person 3 times.

(b) Compute the expectation of X.

$$E(X) = 0 \times \frac{1}{6} + 1 \times \frac{1}{2} + 2 \times \frac{3}{10} + 3 \times \frac{1}{30}$$

$$= 1\frac{1}{5}$$

(c) What is the probability that more than half of the team are from Stanford?

$$P(X = 0 \text{ or } 1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$$

4. At a coffee shop the owner sells a random amount X of coffee each hour. Suppose that X (measured in pounds) has the density function

$$f(x) = \begin{cases} 6x^2 - 4x^3 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The owner buys x pounds of coffee for a price of $2x + 1$ dollars and he sells the same amount for $4x$ dollars. What is his expected profit for a given hour?

Profit function $p(x) = 4x - (2x+1)$
$= 2x - 1$

Expected profit is

$$E(p(x)) = \int_0^1 (2x-1)(6x^2 - 4x^3)\,dx$$

$$= \int_0^1 (12x^3 - 8x^4 - 6x^2 + 4x^3)\,dx$$

$$= \left[3x^4 - \frac{8}{5}x^5 - 2x^3 + x^4\right]_0^1$$

$$= 3 - \frac{8}{5} - 2 + 1$$

$$= \frac{2}{5}$$

Note: $E(X) = .7$, and $p(.7) = \frac{2}{5}$, but this is just by chance. In general, $E(p(x)) \neq p(E(X))$. They are only equal if p happens to be a linear function, as it is here.

5. A certain component in a (shoddy) computer typically fails 40% of the time, causing the computer to break. To counteract this appalling problem, a hacker decides to install n copies of the component in parallel, in such a way that the computer only breaks if all n components fail at the same time. Assume that component failures are independent of each other.

(a) Find the probability that the computer does not break if $n = 3$.

$$
\begin{aligned}
P(\text{Not break}) &= 1 - P(\text{break}) \\
&= 1 - (.4)(.4)(.4) \\
&= .936
\end{aligned}
$$

(b) Find the smallest value of n that should be chosen to ensure that the probability that the computer does not break is at least 98%.

For general n,

$$
\begin{aligned}
P(\text{Not break}) &= 1 - P(\text{break}) \\
&= 1 - (.4)^n
\end{aligned}
$$

$n = 3 \longrightarrow .936$ ×

$n = 4 \longrightarrow .9744$ ×

$n = 5 \longrightarrow .98976$ ✓

So $n = 5$ is good enough

6. (a) The joint probability density function of random variables V and W is given by the formula

$$f(v,w) = \begin{cases} v+w & \text{if } 0 \le v \le 1 \text{ and } 0 \le w \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(V^2W)$.

$$E(V^2W) = \int_0^1\int_0^1 v^2 w(v+w)\,dv\,dw$$

$$= \int_0^1\int_0^1 (v^3w + v^2w^2)\,dv\,dw$$

$$= \int_0^1 \left[\frac{v^4w}{4} + \frac{v^3w^2}{3}\right]_0^1 dw$$

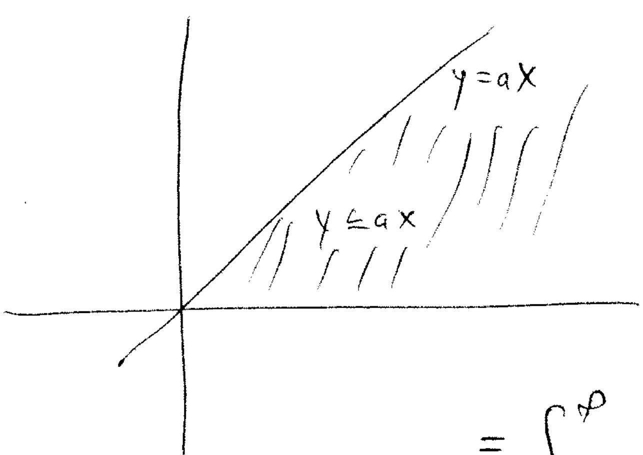
$$= \int_0^1 \left[\frac{w}{4} + \frac{w^2}{3}\right]_0^1$$

$$= \left[\frac{w^2}{8} + \frac{w^3}{9}\right]_0^1 = \frac{1}{8} + \frac{1}{9} = \frac{17}{72}$$

(b) X and Y are two independent random variables that both only take positive values. Show that for each $a > 0$ the value of the distribution function of Y/X at a is

$$F_{Y/X}(a) = \int_0^\infty F_Y(ax) f_X(x)\,dx$$

where $f_X(x)$ is the density function of X and $F_Y(y)$ is the distribution function of Y.



$$F_{\frac{Y}{X}}(a) = P\left(\frac{Y}{X} \le a\right)$$

$$= P(Y \le aX)$$

$$= \iint_{\substack{\text{shaded region} \\ \text{on left}}} f(x,y)\,dA$$

$$= \int_{x=0}^{\infty}\int_{y=0}^{ax} f_X(x) f_Y(y)\,dy\,dx$$

$$= \int_0^\infty f_X(x)\left(\underbrace{\int_0^{ax} f_Y(y)\,dy}_{F_Y(ax)}\right) dx$$

$$= \int_0^\infty F_Y(ax) f_X(x)\,dx.$$

7. When Baltimore Ravens' running back Ray Rice rushes, he advances a number of yards that has mean 4 and standard deviation 1/2.

   (a) Suppose that Rice rushes three times in a row. Let X be the number of yards he advances in total. Assuming that the three rushes are independent of each other, calculate the expectation and variance of X.

X_i = yards on i^{th} rush; $E(X_i) = 4$

$Var(X_i) = (\frac{1}{2})^2 = \frac{1}{4}$

$$X = X_1 + X_2 + X_3$$

$$E(X) = E(X_1) + E(X_2) + E(X_3) = 4 + 4 + 4 = 12$$

$$Var(X) = Var(X_1) + Var(X_2) + Var(X_3) = \frac{3}{4}$$

Note: If Y is the amount gained in one rush, then the amount gained in 3 rushes is not $3Y$. The three rushes are independent, so we have to use $X = X_1 + X_2 + X_3$.

   (b) Use Tcebychev's inequality to find a number p such that the probability that Rice gains between 10 and 14 yards on three successive rushes is at least p.

$$P(10 \leq X \leq 14) = P(|X - 10| \leq 2)$$

$$\geq 1 - \frac{Var(X)}{2^2} \quad \text{(Tchebychev)}$$

$$= 1 - \frac{\frac{3}{4}}{4}$$

$$= \frac{13}{16}.$$

So we can take $p = \frac{13}{16}$.