MATH 30440, SEC 01, SPRING 2010

HOMEWORK 1 SOLUTIONS

3.1) With replacement,

$$S = \{rr, rg, rb, gr, gg, gb, br, bg, bb\}$$

("rg" meaning red first, green second, etc.)

Without replacement,

$$S = \{rg, rb, gr, gb, br, bg\}$$

3.3) a) $EF = \{7\}$

b) $E \cup FG = \{1, 3, 4, 5, 7\}$

c) $EG^c = \{3, 5, 7\}$

3.5) a) $16 \; (= 2 \times 2 \times 2 \times 2)$

b) $\{(1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1),$
$(0,0,1,1), (0,1,1,1), (1,0,1,1)\}$

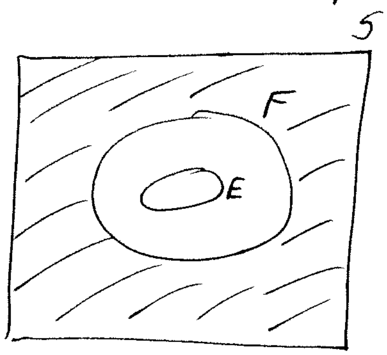
c) $2 \times 2 = 4$ (2 choices for state of comp 2, 2 choices for state of comp 4)

3.6) a) EF^cG^c

b) EF^cG

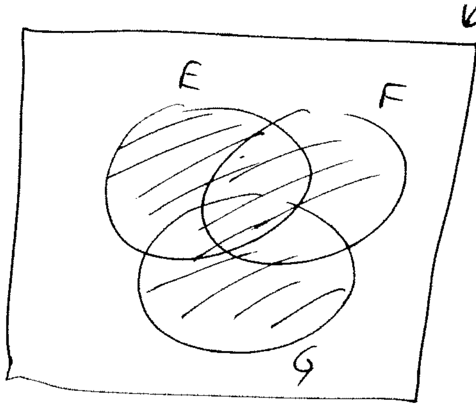
c) $E \cup F \cup G$

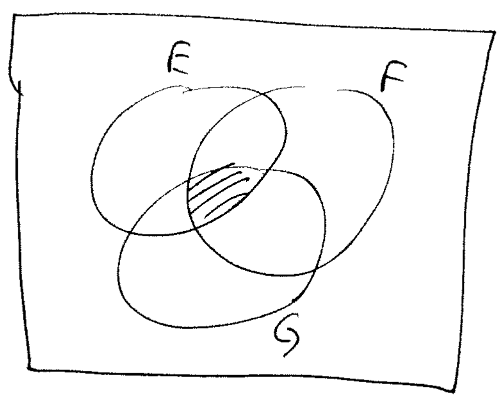
3.8) b) Here's the picture representing $E \subset F$:



The shaded area is F^c; it has nothing in common with E, and so is contained in E^c

d) Both $(E \cup F) \cup G$ and $E \cup (F \cup G)$ are the shaded area below:





Both $(EF)G$ and $E(FG)$ are the shaded area above

3.10) If $E \subset F$ then

$F = E \cup E^c F$, and these are disjoint.

So $P(F) = P(E) + P(E^c F)$

$\geq P(E)$, since $P(E^c F) \geq 0$.

3.11) By induction on n.

For $n=2$: $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2)$

$\leq P(E_1) + P(E_2)$

(Since $P(E_1 E_2) \geq 0$)

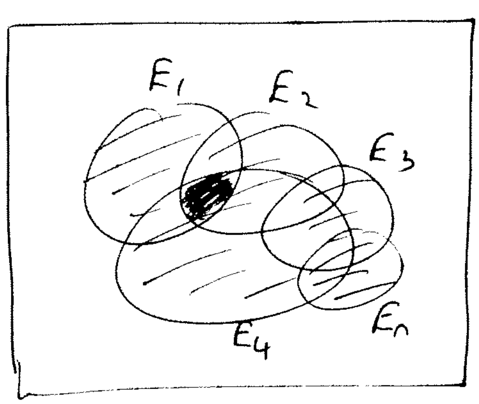
For $n > 2$: $P(E_1 \cup \ldots \cup E_n) =$

$P((E_1 \cup \ldots \cup E_{n-1}) \cup E_n) \leq$

$P(E_1 \cup \ldots \cup E_{n-1}) + P(E_n) \leq$ (base case)

$P(E_1) + \ldots + P(E_{n-1}) + P(E_n)$. ($n-1$ case)

Pictorially:



Both sides cover the shaded region; but rhs covers some points more than once (eg, heavily shaded region counted 3 times).

3.12) $P(E \cup F) = P(E) + P(F) - P(EF)$

Since $P(E \cup F) \leq 1$, get

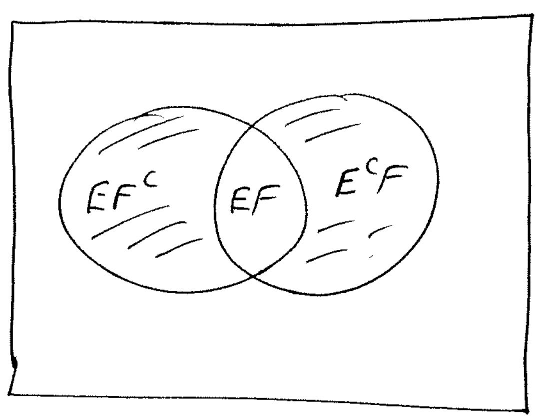
$1 \geq P(E) + P(F) - P(EF)$, or

$P(EF) \geq P(E) + P(F) - 1$.

With $P(E) = .9$ and $P(F) = .9$,

$P(EF) \geq .9 + .9 - 1 = .8$.

3.14)



$P(\text{Exactly one of } E, F) =$

$$P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$$

$$= [P(EF^c) + P(EF)] + [P(E^cF) + P(EF)] - 2P(EF)$$

$$= P(EF^c \cup EF) + P(E^cF \cup EF) - 2P(EF)$$

$$= P(E) + P(F) - 2P(EF).$$

3.15) $\binom{9}{3} = \frac{9!}{3!\,6!} = \frac{9.8.7.6.5.4.3.2.1}{3.2.1.6.5.4.3.2.1} = 84$

$\binom{9}{6} = 84$

$\binom{7}{2} = 21$

$\binom{7}{5} = 21$

$\binom{10}{7} = 120$

3.17) Counting argument : Want to select r objects from among n, where $n-1$ of the n are red and one is blue.

One way : just choose r objects, regardless of colour : $\binom{n}{r}$ ways

Another way : <u>either</u> commit to choosing the blue object : $\binom{n-1}{r-1}$ ways, since remaining $r-1$ objects must be chosen from among the $n-1$ reds or commit to not choosing the blue object: $\binom{n-1}{r}$ ways, since all r objects must be chosen from among the $n-1$ reds.

Gives $\binom{n-1}{r-1} + \binom{n-1}{r}$ total ways

Conclusion: Since either way we get all choices of r objects,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Algebra argument:

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right]$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{n}{(n-r)r}\right]$$

$$= \frac{n!}{r!\,(n-r)!}$$

$$= \binom{n}{r}.$$