MATH 30440, SEC 01, SPRING 2010

HOMEWORK 4 SOLUTIONS

4.21) $E(X) = 1 \times .5 + 2 \times .2\dot{7} + 3 \times .1389$

$+ 4 \times .0595 + 5 \times .0198$

$+ 6 \times .004 \quad = 1.833$

4.24) $\text{Profit} = \begin{cases} \text{Premium} & \bar{w} \text{ prob } 1-p \\ \text{Premium} - A & \bar{w} \text{ prob } p \end{cases}$

Expected profit $= \text{Premium} - pA$

Want this to be $.1A$, so

$\text{Premium} = (p + .1)A$

4.26) $P(\text{lasts 2 games}) = pp + (1-p)(1-p)$ [AA, BB]

$= 1 - 2p + 2p^2$

$P(\text{last 3 games}) = 1 - P(\text{lasts 2})$

$= 2p - 2p^2.$

Expected length is

$$2\left[1-2p+2p^2\right]+3\left[2p-2p^2\right]$$

$$= 2+\cancel{4}2p-2p^2$$

Derivative of this is $2-4p$, which $=0$ when $p=\frac{1}{2}$, so that is the p that maximizes the length

[Max value is $2\frac{1}{2}$]

4.28) First, check what a is by integrating density from 0 to ∞ and setting equal to 1:

$$\int_0^\infty a^2 x e^{-ax}\,dx = \left[a^2 x \cdot \frac{-e^{-ax}}{a}\right]_0^\infty + \int_0^\infty a e^{-ax}\,dx$$

$u = a^2 x \quad dv = e^{-ax}\,dx$

$du = a^2\,dx \quad v = \frac{-e^{-ax}}{a}$

$$= 0 + \left[-e^{-ax}\right]_0^\infty$$

$$= 1$$

Since integral is always 1, a can take any value, so our final answer will be in terms of a.

$$E(X) = \int_0^\infty \underbrace{a^2 x^2}_{v} \underbrace{e^{-ax}}_{dv} \, dx$$

$$= \left[a^2 x^2 \times \frac{-e^{-ax}}{a} \right]_0^\infty - \int_0^\infty \frac{-e^{-ax}}{a} 2xa^2 \, dx$$

$$= \int_0^\infty 2axe^{-ax} \, dx$$

$= \frac{2}{a}$ [It's twice the integral will just did to see what the value of a might be, divided by a.]

4.29a) Refering back to 4.11) we know that the density of

$\text{Max}\{X_1, \dots, X_n\}$ is $f(x) = \begin{cases} nx^{n-1} & 0 \le x \le 1 \\ 0 & \text{owise} \end{cases}$

So $E(\text{Max}) = \int_0^1 xnx^{n-1} \, dx$

$$= \left[\frac{n}{n+1} x^{n+1} \right]_0^1$$

$$= \frac{n}{n+1} = 1 - \frac{1}{n+1}.$$

4.31) Expected cost $= \int_0^2 \text{Cost}(x) f(x)\,dx$

$$= \int_0^2 (40 + 30\sqrt{x}) \tfrac{1}{2}\,dx$$

$$= \left[20x + 10x^{\frac{3}{2}} \right]_0^2$$

$$= 40 + 20\sqrt{2}.$$

$$= 68.284\ldots$$

4.3~~4~~a) Want M so that

$$\int_0^M e^{-x}\,dx = \tfrac{1}{2}$$

ie, $\left[-e^{-x}\right]_0^M = \tfrac{1}{2}$

ie, $1 - e^{-M} = \tfrac{1}{2}$

ie, $M = \ln 2$

4.40) The variance is minimized when as much of the mass function is as close as possible to the mean

$$p_1 = 0, \quad p_2 = 1, \quad p_3 = 0$$

(Variance 0)

The variance is maximized when as much of the mass function is as far from the mean as possible

$$p_1 = \frac{1}{2}, \quad p_2 = 0, \quad p_3 = \frac{1}{2}$$

(Variance 1)

4.41) Binomial trial with $n = 3$, $p = \frac{1}{2}$.

$$\text{Mean} = 0 \times \left(\tfrac{1}{2}\right)^3 + 1 \times 3\left(\tfrac{1}{2}\right)^3 + 2 \times 3\left(\tfrac{1}{2}\right)^3 + 3 \times \left(\tfrac{1}{2}\right)^3$$

$$= 1.5$$

$$\text{Variance} = (0-1.5)^2\left(\tfrac{1}{2}\right)^3 + (1-1.5)^2\left(3\left(\tfrac{1}{2}\right)^3\right) + (2-1.5)^2\left(3\left(\tfrac{1}{2}\right)^3\right) + (3-1.5)^2\left(\tfrac{1}{2}\right)^3$$

$$= .75$$

4.43) a)

$$E(X) = \int x f(x) dx$$

$$= \int_8^9 x(x-8) dx + \int_9^{10} x(10-x) dx$$

$$= \mu$$

$$E(X^2) = \int_8^9 x^2(x-8) dx + \int_9^{10} x^2(10-x) dx$$

$Var(X)$ is then computed as $E(X^2) - \mu^2$

b)

$$E(\text{profit}) = \int \text{profit}(x) f(x) dx$$

$$= \int_8^{8.25} \left(0 - \left(\frac{x}{15} + .35\right)\right)(x-8) dx$$

$$+ \int_{8.25}^9 \left(2 - \left(\frac{x}{15} + .35\right)\right)(x-8) dx$$

$$+ \int_9^{10} \left(2 - \left(\frac{x}{15} + .35\right)\right)(10-x) dx$$

4.45) a) Marginal of X_1:

$$P(X_1 = 0) = \frac{1}{8} + \frac{1}{16} = \frac{3}{16}$$

$$P(X_1 = 1) = \frac{2}{16}$$

$$P(X_1 = 2) = \frac{5}{16}$$

$$P(X_1 = 3) = \frac{6}{16}$$

Marginal of X_2:

$$P(X_2 = 1) = \frac{1}{8} + \frac{1}{16} + \frac{3}{16} + \frac{1}{8} = \frac{1}{2}$$

$$P(X_2 = 2) = \frac{1}{2}$$

b) $E(X_1) = \frac{30}{16}$

$E(X_1^2) = \frac{76}{16}$

$Var(X_1) = \frac{316}{256}$

$E(X_2) = \frac{3}{2}$

$E(X_2^2) = \frac{5}{2}$

$Var(X_2) = \frac{1}{4}$

$E(X_1 X_2) = \frac{47}{16}$

$E(X_1)E(X_2) = \frac{45}{16}$

$Cov(X_1, X_2) = \frac{2}{16} = \frac{1}{8}$

5.3) $P(\text{Exactly } 7 \text{ of } 10) = \binom{10}{7}(.7)^7(.3)^3$

$= .2668$

5.5) $P(4 \text{ engine plane operates})$

$= P(2, 3 \text{ or } 4 \text{ engines function})$

$= \binom{4}{2}p^2(1-p)^2 + \binom{4}{3}p^3(1-p) + \binom{4}{4}p^4$

Easier: $1 - P(0 \text{ or } 1)$

$= 1 - \binom{4}{0}(1-p)^4 - \binom{4}{1}p(1-p)^3$

$= 1 - (1-p)^4 - 4p(1-p)^3$

$P(2 \text{ engine plane operates})$

$= 1 - P(0) = 1 - (1-p)^2$

Want p so that $1-(1-p)^4 - 4p(1-p)^3 > 1-(1-p)^2$

ie $(1-p)^2 + 4p(1-p) < 1$

ie $2p - 3p^2 < 0$

ie $p > \frac{2}{3}$.