Math 30530 — Introduction to Probability

Fall 2009 second mid-term exam

November 18, 2009

Name: SOLUTIONS **Instructor:** David Galvin

This examination contains 6 problems on 7 pages. It is open-book, open-notes. You may use a calculator. **Show all your work** on the paper provided. The honor code is in effect for this examination.

Scores

Question	Score	Out of
1		15
2		15
3		15
4		15
5		15
6		15
Total		90

GOOD LUCK !!!

1. A box contains a large number of nails whose lengths are normally distributed with mean 4 cm and standard deviation .1 cm.

   10 nails are chosen at random from the batch. What is the probability that exactly 5 of them will be between 3.9 cm and 4.1 cm in length?

Let L = length of randomly chosen nail.

$L = N(4, (.1)^2)$

Standard normal

$P(3.9 \le L \le 4.1) = P(-1 \le Z \le 1)$

$= .8413 - .1587$

$= .6826$

Let X = # from among 10 with right length

$X = \text{Binomial}(10, .6826)$

$P(X = 5) = \binom{10}{5}(.6826)^5(.3174)^5$

$= .12\ldots$

2. Dr. G. teaches on MWF 1.55-2:45 pm. On Mondays he usually makes a mistake, on average, every 10 minutes. On Wednesdays he mostly gets his act together, but still adds things incorrectly once about every 25 minutes. By Friday he gets tired and, on average, misspells a word eight times uring the lecture period. On each day, the number of mistakes he makes is modeled by a Poisson random variable.

   If a randomly chosen lecture (equally likely to be M, W or F) happened to be surprisingly error free, what is the probability that it was delivered on Friday?

$M = \{\text{No mistakes}\}$

$P(M \mid \text{Monday}) = e^{-5}$ (Poisson with $\lambda = 5$)

$P(M \mid \text{Wed}) = e^{-2}$ (" " $\lambda = 2$)

$P(M \mid \text{Fri}) = e^{-8}$ (" " $\lambda = 8$)

[Using 50 minute lecture period as unit]

By Bayes,

$$P(\text{Fri} \mid M) = \frac{P(M \mid \text{Fri})P(\text{Fri})}{P(M \mid \text{Fri})P(\text{Fri}) + P(M \mid \text{Wed})P(\text{Wed}) + P(M \mid \text{Mon})P(\text{Mon})}$$

$$= \frac{e^{-8}}{e^{-8} + e^{-2} + e^{-5}}$$

$$= .0024\ldots$$

3. During the slack time from 1pm to 5pm, customers arrive at Reckers at a constant rate of 10 per hour. A customer has just arrived. Let T be the time until the next customer arrives.

   (a) Using an <u>exponential</u> random variable to model T, calculate the probability that it will be more than 10 minutes until the next customer arrives.

Using minutes as units, $T = \text{exponential}\left(\frac{1}{6}\right)$

$$P(T > 10) = \int_{10}^{\infty} \frac{1}{6} e^{-\frac{x}{6}}\,dx$$

$$= \left[-e^{-\frac{x}{6}}\right]_{10}^{\infty} = e^{-\frac{10}{6}} = .188\ldots$$

   (b) Suppose that 5 minutes have passed, and no customer has arrived. Calculate the probability that it will be more than a further 10 minutes until the next customer arrives.

By memorylessness,

$$P(T > 5+10 \mid T > 5) = P(T > 10) = .188\ldots$$

   (c) Write down an integral whose value is the probability that the third customer to arrive after the one who has just come in comes in sometime between 25 and 35 minutes from now.

X = arrival time of third customer.

$= \text{Gamma}, \; n = 3, \; \lambda = \frac{1}{6}$

$$P(25 \le X \le 35) = \int_{25}^{35} \frac{1}{6} e^{-\frac{x}{6}} \frac{\left(\frac{x}{6}\right)^2}{2!}\,dx$$

4. A random variable X has density function given by

$$f(x) = \begin{cases} \frac{3}{x^4} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $E(X)$.

$$E(X) = \int_1^\infty \frac{3x}{x^4}\,dx = \left[-\frac{3}{2}x^{-2}\right]_1^\infty = \frac{3}{2}$$

(b) Compute the distribution function $F_{X^2}(t)$ of X^2. (Give its value for all t.)

For $t < 1$, $P(X^2 \leq t) = 0$, So $F_{X^2}(t) = 0$

For $t \geq 1$, $P(X^2 \leq t) = P(X \leq \sqrt{t})$

$$= \int_1^{\sqrt{t}} \frac{3}{x^4}\,dx = \left[-x^{-3}\right]_1^{\sqrt{t}} = 1 - t^{-\frac{3}{2}}$$

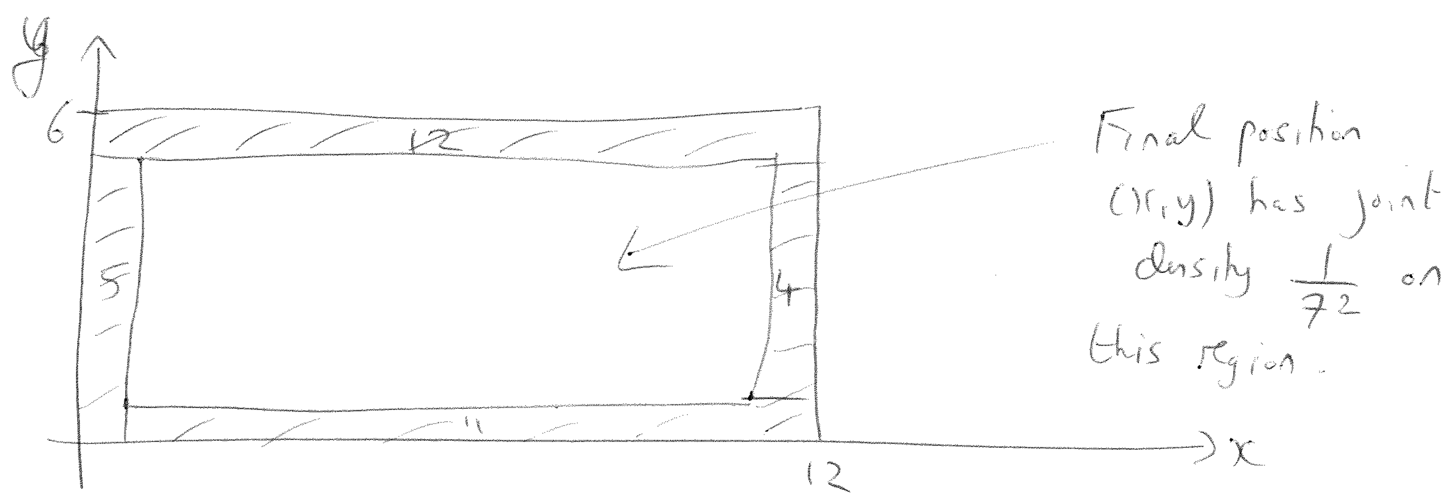
So $F_{X^2}(t) = 1 - t^{-\frac{3}{2}}$ for $t \geq 1$

(c) Compute the density function $f_{X^2}(x)$ of X^2.

Density = derivative of distribution

$$f_{X^2}(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{3}{2}\cancel{X}^{-\frac{5}{2}} & \text{if } x \geq 1 \end{cases}$$

5. A billiard table measures 12 feet by 6 feet. The spot at which a ball comes to rest after being hit very hard is randomly distributed across the whole board. What is the probability that the ball will come to rest within 1 foot of one of the edges of the table?



Area of table $= 72$ sq feet

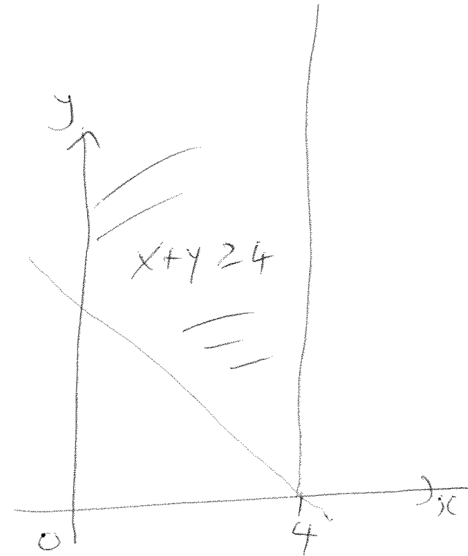
Area within one foot of edge $= 12 + 5 + 11 + 4$

$= 32$

$$P(\text{Within one foot of an edge}) = \frac{32}{72} = .44\ldots$$

6. The joint density of a pair of random variables X, Y is

$$f(x,y) = \begin{cases} Cxe^{-4y} & \text{if } 0 \le x \le 4 \text{ and } 0 \le y \le \infty \\ 0 & \text{otherwise.} \end{cases}$$



(a) Find C.

$$\int_0^4 \int_0^\infty Cxe^{-4y}\,dy\,dx = C\int_0^4 x\,dx \int_0^\infty e^{-4y}\,dy$$

$$= C\left[\frac{x^2}{2}\right]_0^4 \left[\frac{e^{-4y}}{-4}\right]_0^\infty$$

$$= 8C \times \frac{1}{4} = 2C$$

Should $= 1$, so $C = \frac{1}{2}$

(b) Find the marginal density of X.

$$f_X(x) = \int_0^\infty \frac{x}{2} e^{-4y}\,dy = \frac{x}{2}\left[\frac{e^{-4y}}{-4}\right]_0^\infty$$

$$= \frac{x}{8}, \quad 0 \le x \le 4,$$

$f_X(x) = 0$ otherwise

(c) Write down an integral whose value is the probability that $X + Y \ge 4$.

$$\int_{x=0}^{4} \int_{y=4-x}^{\infty} \frac{x}{2} e^{-4y}\,dy\,dx$$