MATH 30530, Fall 2009, E2 Review Solutions

Page 228:

1. $\sum_{i=12}^{20}\binom{20}{i}(0.25)^i(0.75)^{20-i} = 0.0009.$

2. $N(t)$, the number of customers arriving at the post office at or prior to t is a Poisson process with $\lambda = 1/3$. Thus

$$P\big(N(30) \le 6\big) = \sum_{i=0}^{6} P\big(N(30) = i\big) = \sum_{i=0}^{6} \frac{e^{-(1/3)30}\big[(1/3)30\big]^i}{i!} = 0.130141.$$

7. $\sum_{i=7}^{12} \frac{\binom{160}{i}\binom{200}{12-i}}{\binom{360}{12}} = 0.244.$

10. Suppose that 5% of the items are defective. Under this hypothesis, there are $500(0.05) = 25$ defective items. The probability of two defective items among 30 items selected at random is

$$\frac{\binom{25}{2}\binom{475}{28}}{\binom{500}{30}} = 0.268.$$

Therefore, under the above hypothesis, having two defective items among 30 items selected at random is quite probable. The shipment should not be rejected.

14. Call a customer a "success," if he or she will make a purchase using a credit card. Let E be the event that a customer entering the store will make a purchase. Let F be the event that the customer will use a credit card. To find p, the probability of success, we use the law of multiplication:

$$p = P(EF) = P(E)P\big(F \mid E\big) = (0.30)(0.85) = 0.255.$$

The random variable X is binomial with parameters 6 and 0.255. Hence

$$P\big(X = i\big) = \binom{6}{i}(0.255)^i\big(1 - 0.255\big)^{6-i}, \quad i = 0, 1, \dots, 6.$$

Clearly, $E(X) = np = 6(0.255) = 1.53$ and

$$\mathrm{Var}(X) = np(1 - p) = 6(0.255)(1 - 0.255) = 1.13985.$$

21. **(a)** $(0.999)^{999}(0.001)^1 = 0.000368.$ **(b)** $\binom{2999}{2}(0.001)^3(0.999)^{2997} = 0.000224.$

22. Let X be the number of children having the disease. We have that the desired probability is

$$P(X = 3 \mid X \ge 1) = \frac{P(X = 3)}{P(X \ge 1)} = \frac{\binom{5}{3}(0.23)^3(0.77)^2}{1 - (0.77)^5} = 0.0989.$$

24. Let n be the desired number of seeds to be planted. Let X be the number of seeds which will germinate. We have that X is binomial with parameters n and 0.75. We want to find the smallest n for which

$$P(X \geq 5) \geq 0.90.$$

or, equivalently,

$$P(X < 5) \leq 0.10.$$

That is, we want to find the smallest n for which

$$\sum_{i=0}^{4} \binom{n}{i} (0.75)^i (.25)^{n-i} \leq 0.10.$$

By trial and error, as the following table shows, we find that the smallest n satisfying $P(X < 5) \leq 0.10$ is 9. So at least nine seeds is to be planted.

n	$\sum_{i=0}^{4} \binom{n}{i} (0.75)^i (.25)^{n-i}$
5	0.7627
6	0.4661
7	0.2436
8	0.1139
9	0.0489

26. The size of a seed is a tiny fraction of the size of the area. Let us divide the area up into many small cells each about the size of a seed. Assume that, when the seeds are distributed, each of them will land in a single cell. Accordingly, the number of seeds distributed will equal the number of nonempty cells. Suppose that each cell has an equal chance of having a seed independent of other cells (this is only approximately true). Since λ is the average number of seeds per unit area, the expected number of seeds in the area, A, is λA. Let us call a cell in A a "success" if it is occupied by a seed. Let n be the total number of cells in A and p be the probability that a cell will contain a seed. Then X, the number of cells in A with seeds is a binomial random variable with parameters n and p. Using the formula for the expected number of successes in a binomial distribution ($= np$), we see that $np = \lambda A$ and $p = \lambda A/n$. As n goes to infinity, p approaches zero while np remains finite. Hence the number of seeds that fall on the area A is a Poisson random variable with parameter λA and

$$P(X = i) = \frac{e^{-\lambda A}(\lambda A)^i}{i!}.$$

Page 258 :

1. Let F be the distribution function of Y. Clearly, $F(y) = 0$ if $y \leq 1$. For $y > 1$,

$$F(y) = P\Big(\frac{1}{X} \leq y\Big) = P\Big(X \geq \frac{1}{y}\Big) = \frac{1 - \frac{1}{y}}{1 - 0} = 1 - \frac{1}{y}.$$

So

$$f(y) = F'(y) = \begin{cases} 1/y^2 & y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

2. $E(X) = \displaystyle\int_1^\infty x \cdot \frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx = -\frac{2}{x}\Big|_1^\infty = 2,$

$E(X^2) = \displaystyle\int_1^\infty x^2 \cdot \frac{2}{x^3}\,dx = 2\ln x\Big|_1^\infty = \infty$. So $\mathrm{Var}(X)$ does not exist.

4. We have that

$$P(-2 < X < 1) = \int_{-2}^{1} \frac{e^{-|x|}}{2}\,dx = \frac{1}{2}\Big[\int_{-2}^{0} e^{x}\,dx + \int_{0}^{1} e^{-x}\,dx\Big]$$
$$= 1 - \frac{1}{2e} - \frac{1}{2e^2} = 0.748.$$

6. The set of possible values of X is $A = [1, 2]$. Let $h\colon [1, 2] \to \mathbf{R}$ be defined by $h(x) = e^x$. The set of possible values of e^X is $B = [e, e^2]$; the inverse of h is $g(y) = \ln y$, where $g'(y) = 1/y$. Therefore,

$$f_Y(y) = \frac{4(\ln y)^3}{15}|g'(y)| = \frac{4(\ln y)^3}{15y}, \quad y \in [e, e^2].$$

Applying the same procedure to Z and W, we obtain

$$f_Z(z) = \frac{4(\sqrt{z}\,)^3}{15}\left|\frac{1}{2\sqrt{z}}\right| = \frac{2z}{15}, \quad z \in [1, 4].$$

$$f_W(w) = \frac{2(1+\sqrt{w}\,)^3}{15\sqrt{w}} \quad w \in [0, 1].$$

9. Clearly $\sum_{i=1}^{n} \alpha_i f_i \geq 0$. Since

$$\int_{-\infty}^{\infty}\Big(\sum_{i=1}^{n}\alpha_i f_i\Big)(x)\,dx = \sum_{i=1}^{n}\alpha_i \int_{-\infty}^{\infty} f_i(x)\,dx = \sum_{i=1}^{n}\alpha_i = 1,$$

$\sum_{i=1}^{n} \alpha_i f_i$ is a probability density function.

11. Let X be the lifetime of a random light bulb. The probability that it lasts over 1000 hours is

$$P(X > 1000) = \int_{1000}^{\infty} \frac{5\times 10^5}{x^3}\,dx = 5\times 10^5\Big[-\frac{1}{2x^2}\Big]_{1000}^{\infty} = \frac{1}{4}.$$

Thus the probability that out of six such light bulbs two last over 1000 hours is

$$\binom{6}{2}\Big(\frac{1}{4}\Big)^2\Big(\frac{3}{4}\Big)^4 \approx 0.3$$

Page 3 of :

2. Let X be the weight of a randomly selected women from this community. The desired quantity is

$$P(X > 170 \mid X > 140) = \frac{P(X > 170)}{P(X > 140)} = \frac{P\Big(Z > \dfrac{170-130}{20}\Big)}{P\Big(Z > \dfrac{140-130}{20}\Big)}$$

$$= \frac{P(Z > 2)}{P(Z > 0.5)} = \frac{1-\Phi(2)}{1-\Phi(0.5)} = \frac{1-0.9772}{1-0.6915} = 0.074.$$

3. Let X be the number of times the digit 5 is generated; X is binomial with parameters $n = 1000$ and $p = 1/10$. Thus $np = 100$ and $\sqrt{np(1-p)} = \sqrt{90} = 9.49$. Using normal approximation and making correction for continuity,

$$P(X \le 93.5) = P\Big(Z \le \frac{93.5 - 100}{9.49}\Big) = P(Z \le -0.68) = 1 - \Phi(0.68) = 0.248.$$

4. The given relation implies that

$$1 - e^{-2\lambda} = 2\big[(1 - e^{-3\lambda}) - (1 - e^{-2\lambda})\big].$$

This is equivalent to

$$3e^{-2\lambda} - 2e^{-3\lambda} - 1 = 0,$$

or, equivalently,

$$\big(e^{-\lambda} - 1\big)^2\big(2e^{-\lambda} + 1\big) = 0.$$

The only root of this equation is $\lambda = 0$ which is not acceptable. Therefore, it is not possible that X satisfy the given relation.

6. Note that $\lim_{x \to 0} x \ln x = 0$; so

$$E(-\ln X) = \int_0^1 (-\ln x)\,dx = \Big[x - x \ln x\Big]_0^1 = 1.$$

8. If $\alpha < 0$, then $\alpha + \beta < \beta$; therefore,

$$P(\alpha \le X \le \alpha + \beta) = P(0 \le X \le \alpha + \beta) \le P(0 \le X \le \beta).$$

If $\alpha > 0$, then $e^{-\lambda\alpha} < 1$. Thus

$$\begin{aligned} P(\alpha \le X \le \alpha + \beta) &= \big[1 - e^{-\lambda(\alpha+\beta)}\big] - \big(1 - e^{-\lambda\alpha}\big) \\ &= e^{-\lambda\alpha}\big(1 - e^{-\lambda\beta}\big) < 1 - e^{-\lambda\beta} = P(0 \le X \le \beta). \end{aligned}$$

9. We are given that $1/\lambda = 1.25$; so $\lambda = 0.8$. Let X be the time it takes for a random student to complete the test. Since $P(X > 1) = e^{-(0.8)1} = e^{-0.8}$, the desired probability is

$$1 - \big(e^{-0.8}\big)^{10} = 1 - e^{-8} = 0.99966.$$

10. Note that

$$f(x) = ke^{-[x-(3/2)]^2 + 17/4} = ke^{17/4} \cdot e^{-[x-(3/2)]^2}.$$

Comparing this with the probability density function of a normal random variable with mean 3/2, we see that $\sigma^2 = 1/2$ and $ke^{17/4} = 1/(\sigma\sqrt{2\pi})$. Therefore,

$$k = \frac{1}{\sigma\sqrt{2\pi}} e^{-17/4} = \frac{1}{\pi} e^{-17/4}.$$

16. Let X be the time until the 91st call is received. X is a gamma random variable with parameters $r = 91$ and $\lambda = 23$. The desired probability is

$$P(X \geq 4) = \int_4^\infty \frac{23e^{-23x}(23x)^{91-1}}{\Gamma(91)}\,dx$$

$$= 1 - \int_0^4 \frac{23e^{-23x}(23x)^{91-1}}{90!}\,dx$$

$$= 1 - \frac{23^{91}}{90!}\int_0^4 x^{90}e^{-23x}\,dx = 1 - 0.55542 = 0.44458.$$

Page 365 :

1. **(a)** We have that

$$P(XY \leq 6) = p(1,2) + p(1,4) + p(1,6) + p(2,2) + p(3,2)$$
$$= 0.05 + 0.14 + 0.10 + 0.25 + 0.15 = 0.69.$$

(b) First we calculate $p_X(x)$ and $p_Y(y)$, the marginal probability mass functions of X and Y. They are given by the following table.

	x			
y	1	2	3	$p_Y(y)$
2	0.05	0.25	0.15	0.45
4	0.14	0.10	0.17	0.41
6	0.10	0.02	0.02	0.14
$p_X(x)$	0.29	0.37	0.34	

Therefore,

$$E(X) = 1(0.29) + 2(0.37) + 3(0.34) = 2.05;$$
$$E(Y) = 2(0.45) + 4(0.41) + 6(0.14) = 3.38.$$

4. The set of possible values of X and Y, both, is $\{0, 1, 2, 3\}$. Let $p(x, y)$ be their joint probability mass function; then

$$p(x,y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{3-x-y}}{\binom{52}{3}}, \quad 0 \leq x,\ y,\ x+y \leq 3.$$

7. Note that $f(x, y) = \frac{1}{2}y\left(\frac{3}{2}x^2 + \frac{1}{2}\right)$, where $\frac{1}{2}y$, $0 < y < 2$ and $\frac{3}{2}x^2 + \frac{1}{2}$, $0 < x < 1$ are probability density functions. Therefore,

$$f_Y(y) = \frac{1}{2}y, \quad 0 < y < 2,$$

$$f_X(x) = \frac{3}{2}x^2 + \frac{1}{2}, \quad 0 < x < 1.$$

We observe that $f(x, y) = f_X(x)f_Y(y)$. This shows that X and Y are independent random variables and hence $E(XY) = E(X)E(Y)$. This relation can also be verified directly:

$$E(XY) = \int_0^1 \left[\int_0^2 \left(\frac{3}{4}x^3y^2 + \frac{1}{4}xy^2\right) dy\right] dx = \frac{5}{6},$$

$$E(X) = \int_0^1 \left[\int_0^2 \left(\frac{3}{4}x^3y + \frac{1}{4}xy\right) dy\right] dx = \frac{5}{8},$$

$$E(Y) = \int_0^1 \left[\int_0^2 \left(\frac{3}{4}x^2y^2 + \frac{1}{4}y^2\right) dy\right] dx = \frac{4}{3}.$$

Hence

$$E(XY) = \frac{5}{6} = \frac{5}{8} \cdot \frac{4}{3} = E(X)E(Y).$$

11. $f(x, y)$, the joint probability density function of X and Y is given by

$$f(x, y) = \frac{\partial^2}{\partial x\, \partial y}F(x, y) = 4xye^{-x^2}e^{-y^2}, \quad x > 0, \ y > 0.$$

Therefore, by symmetry,

$$P(X > 2Y) + P(Y > 2X) = 2P(X > 2Y) = 2\int_0^\infty \left(\int_{2y}^\infty 4xye^{-x^2}e^{-y^2}\, dx\right) dy = \frac{2}{5}.$$

12. We have that

$$f_X(x) = \int_0^{1-x} 3(x + y)\, dy = -\frac{3}{2}x^2 + \frac{3}{2}, \quad 0 < x < 1,$$

By symmetry,

$$f_Y(y) = -\frac{3}{2}y^2 + \frac{3}{2}, \quad 0 < y < 1.$$

Therefore,

$$P(X + Y > 1/2) = \int_0^{1/2} \left[\int_{(1/2)-x}^{1-x} 3(x + y)\, dy\right] dx + \int_{1/2}^1 \left[\int_0^{1-x} 3(x + y)\, dy\right] dx$$

$$= \frac{9}{64} + \frac{5}{16} = \frac{29}{64}.$$

20. Let $p(x, y)$ be the joint probability mass function of X and Y.

$$p(x, y) = P(X = x,\ Y = y) = (0.90)^{x-1}(0.10)(0.90)^{y-1}(0.10) = (0.90)^{x+y-2}(0.10)^2.$$