MATH30530, Fall 2009, Homework 10 Solutions

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1. Let f be the probability density function of a gamma random variable with parameters r and λ. Then

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}.$$

Therefore,

$$f'(x) = \frac{\lambda^r}{\Gamma(r)}\big[-\lambda e^{-\lambda x} x^{r-1} + e^{-\lambda x}(r-1)x^{r-2}\big] = -\frac{\lambda^{r+1}}{\Gamma(r)} x^{r-2} e^{-\lambda x}\Big(x - \frac{r-1}{\lambda}\Big).$$

This relation implies that the function f is increasing if $x < (r-1)/\lambda$, it is decreasing if $x > (r-1)/\lambda$, and $f'(x) = 0$ if $x = (r-1)/\lambda$. Therefore, $x = (r-1)/\lambda$ is a maximum of the function f. Moreover, since f' has only one root, the point $x = (r-1)/\lambda$ is the only maximum of f.

3. Let $N(t)$ be the number of babies born at or prior to t. $\{N(t) : t \geq 0\}$ is a Poisson process with $\lambda = 12$. Let X be the time it takes before the next three babies are born. The random variable X is gamma with parameters 3 and 12. The desired probability is

$$P(X \geq 7/24) = \int_{7/24}^{\infty} \frac{12e^{-12x}(12x)^2}{\Gamma(3)}\,dx = 864\int_{7/24}^{\infty} x^2 e^{-12x}\,dx.$$

Applying integration by parts twice, we get

$$\int x^2 e^{-12x}\,dx = -\frac{1}{12}x^2 e^{-12x} - \frac{1}{72}xe^{-12x} - \frac{1}{864}e^{-12x} + c.$$

Thus

$$P\Big(X \geq \frac{7}{24}\Big) = 864\Big[-\frac{1}{12}x^2e^{-12x} - \frac{1}{72}xe^{-12x} - \frac{1}{864}e^{-12x}\Big]_{7/24}^{\infty} = 0.3208.$$

Remark: A simpler way to do this problem is to avoid gamma random variables and use the properties of Poisson processes:

$$P\Big(N\Big(\frac{7}{24}\Big) \leq 2\Big) = \sum_{i=0}^{2} P\Big(N\Big(\frac{7}{24}\Big) = i\Big) = \sum_{i=0}^{2} \frac{e^{-(7/24)12}\big[(7/24)12\big]^i}{i!} = 0.3208.$$

4.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{\Gamma(r)}\,dx = \frac{\lambda^r}{\Gamma(r)}\int_0^{\infty} e^{-\lambda x}x^{r-1}\,dx.$$

Let $t = \lambda x$; then $dt = \lambda dx$, so

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \frac{\lambda^r}{\Gamma(r)}\int_0^{\infty} e^{-t}\cdot\frac{t^{r-1}}{\lambda^{r-1}}\cdot\frac{1}{\lambda}\,dx\\ &= \frac{1}{\Gamma(r)}\int_0^{\infty} e^{-t}t^{r-1}\,dt = \frac{1}{\Gamma(r)}\Gamma(r) = 1.\end{aligned}$$

8. (a) Let F be the probability distribution function of Y. For $t \le 0$, $F(t) = P(Z^2 \le t) = 0$. For $t > 0$,

$$F(t) = P(Y \le t) = P(Z^2 \le t) = P\big(-\sqrt{t} \le Z \le \sqrt{t}\big)$$
$$= \Phi(\sqrt{t}) - \Phi(-\sqrt{t}) = \Phi(\sqrt{t}) - \big[1 - \Phi(\sqrt{t})\big] = 2\Phi(\sqrt{t}) - 1.$$

Let f be the probability density function of Y. For $t \le 0$, $f(t) = 0$. For $t > 0$,

$$f(t) = F'(t) = 2 \cdot \frac{1}{2\sqrt{t}} \Phi'\big(\sqrt{t}\big) = \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-t/2} = \frac{1}{\sqrt{2\pi t}} e^{-t/2} = \frac{\frac{1}{2} e^{-t/2}\left(\frac{1}{2}t\right)^{-1/2}}{\Gamma(1/2)},$$

where by the previous exercise, $\sqrt{\pi} = \Gamma(1/2)$. This shows that Y is gamma with parameters $\lambda = 1/2$ and $r = 1/2$.

9. The following solution is an intuitive one. A rigorous mathematical solution would have to consider the sum of two random variables, each being the minimum of n exponential random variables; so it would require material from joint distributions. However, the intuitive solution has its own merits and it is important for students to understand it.

Let the time Howard enters the bank be the origin and let $N(t)$ be the number of customers served by time t. As long as all of the servers are busy, due to the memoryless property of the exponential distribution, $\{N(t) : t \ge 0\}$ is a Poisson process with rate $n\lambda$. This follows because if one server serves at the rate λ, n servers will serve at the rate $n\lambda$. For the Poisson process $\{N(t) : t \ge 0\}$, every time a customer is served and leaves, an "event" has occurred. Therefore, again because of the memoryless property, the service time of the person ahead of Howard begins when the first "event" occurs and Howard's service time begins when the second "event" occurs. Therefore, Howard's waiting time in the queue is the time of the second event of the Poisson process $\{N(t), t \ge 0\}$. This period, as we know, has a gamma distribution with parameters 2 and $n\lambda$.

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1. (a) $\sum_{x=1}^{2}\sum_{y=1}^{2} k(x/y) = 1$ implies that $k = 2/9$.

(b) $p_X(x) = \sum_{y=1}^{2}\big[(2x)/(9y)\big] = x/3, \quad x = 1, 2.$

$p_Y(y) = \sum_{x=1}^{2}\big[(2x)/(9y)\big] = 2/(3y), \quad y = 1, 2.$

(c) $P(X > 1 \mid Y = 1) = \dfrac{p(2,1)}{p_Y(1)} = \dfrac{4/9}{2/3} = \dfrac{2}{3}.$

(d) $E(X) = \displaystyle\sum_{y=1}^{2}\sum_{x=1}^{2} x \cdot \frac{2}{9}\left(\frac{x}{y}\right) = \frac{5}{3}; \qquad E(Y) = \sum_{y=1}^{2}\sum_{x=1}^{2} y \cdot \frac{2}{9}\left(\frac{x}{y}\right) = \frac{4}{3}.$

3. (a) $k(1+1+1+9+4+9) = 1$ implies that $k = 1/25$.

(b)
$$p_X(1) = p(1,1) + p(1,3) = 12/25, \qquad p_X(2) = p(2,3) = 13/25;$$
$$p_Y(1) = p(1,1) = 2/25, \qquad p_Y(3) = p(1,3) + p(2,3) = 23/25.$$

Therefore,

$$p_X(x) = \begin{cases} 12/25 & \text{if } x = 1 \\ 13/25 & \text{if } x = 2, \end{cases} \qquad p_Y(y) = \begin{cases} 2/25 & \text{if } y = 1 \\ 23/25 & \text{if } y = 3. \end{cases}$$

(c) $E(X) = 1 \cdot \dfrac{12}{25} + 2 \cdot \dfrac{13}{25} = \dfrac{38}{25}$; $\quad E(Y) = 1 \cdot \dfrac{2}{25} + 3 \cdot \dfrac{23}{25} = \dfrac{71}{25}$.

6. The following table gives $p(x, y)$, the joint probability mass function of X and Y; $p_X(x)$, the marginal probability mass function of X; and $p_Y(y)$, the marginal probability mass function of Y.

	y						
x	0	1	2	3	4	5	$p_X(x)$
2	1/36	0	0	0	0	0	1/36
3	0	2/36	0	0	0	0	2/36
4	1/36	0	2/36	0	0	0	3/36
5	0	2/36	0	2/36	0	0	4/36
6	1/36	0	2/36	0	2/36	0	5/36
7	0	2/36	0	2/36	0	2/36	6/36
8	1/36	0	2/36	0	2/36	0	5/36
9	0	2/36	0	2/36	0	0	4/36
10	1/36	0	2/36	0	0	0	3/36
11	0	2/36	0	0	0	0	2/36
12	1/36	0	0	0	0	0	1/36
$p_Y(y)$	6/36	10/36	8/36	6/36	4/36	2/36	

8. (a) For $0 \le x \le 7, 0 \le y \le 7, 0 \le x + y \le 7$,

$$p(x, y) = \frac{\dbinom{13}{x}\dbinom{13}{y}\dbinom{26}{7-x-y}}{\dbinom{52}{7}}.$$

For all other values of x and y, $p(x, y) = 0$.

(b) $P(X \ge Y) = \sum_{y=0}^{3} \sum_{x=y}^{7-y} p(x, y) = 0.61107$.

9. **(a)** $f_X(x) = \int_0^x 2\,dy = 2x, \quad 0 \le x \le 1; \quad f_Y(y) = \int_y^1 2\,dx = 2(1-y), \quad 0 \le y \le 1.$

(b) $E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(2x)\,dx = 2/3;$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 2y(1-y)\,dy = 1/3.$$

(c) $P\Big(X < \frac{1}{2}\Big) = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} 2x\,dx = \frac{1}{4},$

$$P(X < 2Y) = \int_0^1 \int_{x/2}^x 2\,dy\,dx = \frac{1}{2},$$

$$P(X = Y) = 0.$$

11. $f_X(x) = \int_0^2 \frac{1}{2} y e^{-x}\,dy = e^{-x}, \quad x > 0; \quad f_Y(y) = \int_0^\infty \frac{1}{2} y e^{-x}\,dx = \frac{1}{2}y, \quad 0 < y < 2.$

12. Let $R = \big\{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\big\}$. Since area$(R) = 1$, $P(X + Y \le 1/2)$ is the area of the region $\big\{(x, y) \in R : x + y \le 1/2\big\}$ which is 1/8. Similarly, $P(X - Y \le 1/2)$ is the area of the region $\big\{(x, y) \in R : x - y \le 1/2\big\}$ which is 7/8. $P(X^2 + Y^2 \le 1)$ is the area of the region $\big\{(x, y) \in R : x^2 + y^2 \le 1\big\}$ which is $\pi/4$. $P(XY \le 1/4)$ is the sum of the area of the region $\big\{(x, y) : 0 \le x \le 1/4,\ 0 \le y \le 1\big\}$ which is 1/4 and the area of the region under the curve $y = 1/(4x)$ from $1/4$ to 1. (Draw a figure.) Therefore,

$$P(XY \le 1/4) = \frac{1}{4} + \int_{1/4}^1 \frac{1}{4x}\,dx \approx 0.597.$$

15. Let X and Y be two randomly selected points from the interval $(0, \ell)$. We are interested in $E\big(|X - Y|\big)$. Since the joint probability density function of X and Y is

$$f(x, y) = \begin{cases} \dfrac{1}{\ell^2} & 0 < x < \ell,\ 0 < y < \ell \\ 0 & \text{elsewhere}, \end{cases}$$

$$\begin{aligned} E\big(|X - Y|\big) &= \int_0^\ell \int_0^\ell |x - y| \frac{1}{\ell^2}\,dx\,dy \\ &= \frac{1}{\ell^2} \int_0^\ell \Big[\int_0^y (y - x)\,dx\Big] dy + \frac{1}{\ell^2} \int_0^\ell \Big[\int_y^\ell (x - y)\,dx\Big] dy \\ &= \frac{\ell}{6} + \frac{\ell}{6} = \frac{\ell}{3}. \end{aligned}$$

16. The problem is equivalent to the following: Two random numbers X and Y are selected at random and independently from $(0, \ell)$. What is the probability that $|X - Y| < X$? Let $S = \big\{(x, y) : 0 < x < \ell,\ 0 < y < \ell\big\}$ and

$$R = \big\{(x, y) \in S : |x - y| < x\big\} = \big\{(x, y) \in S : y < 2x\big\}.$$

The desired probability is the area of R which is $3\ell^2/4$ divided by ℓ^2. So the answer is 3/4. (Draw a figure.)