Math 30530: Introduction to Probability, Fall 2011

Midterm Exam 1

Solutions

1. For any three events A, B and C, say whether each of the statements below are always true or sometimes false/sometimes true. If true, give a proof; if sometimes false/sometimes true, give examples based on the experiment of rolling a die and observing the number that is rolled.

   (a) $P(AB^c) + P(A^cB) \leq 1$.
   Solution: AB^c and A^cB are mutually exclusive, so by one of the axioms of probability we have $P(AB^c) + P(A^cB) = P(AB^c \cup A^cB)$. By another of the axioms, this is always less than or equal to 1.

   (b) $P(AB) \geq P(A \cup B)$.
   Solution: If $A = B$, this is true, since in this case $AB = A \cup B = A$. But if $A \neq B$, then it is usually false. For example, if $A = \{1\}$ and $B = \{2\}$ then $P(AB) = 0$ and $P(A \cup B) = 1/3$.

   (c) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.
   Solution: We know that

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC),$$

   so to show that this is true, we just want to show that $P(AB) + P(AC) + P(BC) - P(ABC) \geq 0$. Since $ABC \subset BC$, we have $P(BC) - P(ABC) \geq 0$ (one of the consequences of the axioms), and by the axioms we have $P(AB) + P(AC) \geq 0$, so indeed $P(AB) + P(AC) + P(BC) - P(ABC) \geq 0$, and the asserted inequality is always true.

2. An ordinary deck of 52 cards is thoroughly shuffled. What is the probability that the four aces will occur consecutively?

   Solution: There are 52! ways to order the 52 cards. To count how many of these ways put the four aces consecutively, we first note that there are 49 places to locate the start of the run of four aces (the first of them can be anywhere between the first and 49th card), then 4! ways to order the four aces, then 48! ways to order the remaining 48 cards, leading to $49 * 4! * 48!$ ways. So the required probability is

   $$\frac{49 * 4! * 48!}{52!}.$$

3. A woman gives birth to a child in the hospital's maternity ward. After a while, the child is brought into the hospital's nursery. Before the child is brought in, there are 5 boys and 10

girls in the nursery. Sometime after the child is brought in, a doctor walks into the nursery, picks a child at random, and notices that it's a boy. What's the probability that the woman gave birth to a boy? (You should assume that when a child is born, the probability that it is a boy is .5 and the probability that it is a girl is .5.)

Solution: Let B be the event that the woman gave birth to a boy, G the event that she gave birth to a girl, and E the event that the doctor observed a boy. We are given:

$$P(B) = .5,\ P(G) = .5,\ P(E|B) = \frac{6}{16},\ \text{and } P(E|G) = \frac{5}{16}.$$

By Bayes' formula,

$$P(B|E) = \frac{P(E|B)P(B)}{P(E|B)P(B) + P(E|G)P(G)} = \frac{\frac{6}{16} * \frac{1}{2}}{\frac{6}{16} * \frac{1}{2} + \frac{5}{16} * \frac{1}{2}} = \frac{6}{11}.$$

4. A fair coin is tossed n times. Let E be the event that both heads and tails occur, and let F be the event that at most one head occurs.

   (a) If $n = 2$, are the events E and F independent? Justify your answer.
   Solution: $E = \{HT, TH\}$, $F = \{HT, TH, TT\}$ and $EF = \{HT, TH\}$, so $P(E) = 1/2$, $P(F) = 3/4$, $P(EF) = 1/2$ and $P(E)P(F) \neq P(E)P(F)$; the events are *not* independent.

   (b) If $n = 3$, are the events E and F independent? Justify your answer.
   Solution: $E = \{HHT, HTH, THH, TTH, THT, HTT\}$, $F = \{HTT, HTH, THH, TTT\}$ and $EF = \{HTT, HTH, THH\}$, so $P(E) = 3/4$, $P(F) = 1/2$, $P(EF) = 3/8$ and $P(E)P(F) = P(E)P(F)$; the events *are* independent.

5. A jar contains 2 blue balls and 2 red balls. A random sample of two balls is taken from the jar, without replacement. Let X be the number of blue balls in the sample.

   (a) Find the probability mass function of X.
   Solution: The possible values for X are 0, 1 and 2. Note that because the drawings are happening *without* replacement, X is *not* binomial! We have

   $$P(X = 0) = \frac{2 * 1}{4 * 3} = \frac{1}{6},\ P(X = 1) = \frac{2 * 2 + 2 * 2}{4 * 3} = \frac{2}{3},\ \text{and } P(X = 2) = \frac{2 * 1}{4 * 3} = \frac{1}{6}.$$

   (b) Find $E(X)$.
   Solution: $E(X) = 0 * (1/6) + 1 * (2/3) + 2 * (1/6) = 1$.

   (c) Find $\text{Var}(X)$.
   Solution: $E(X^2) = 0*(1/6)+1*(2/3)+4*(1/6) = 4/3$, so $\text{Var}(X) = E(X^2)-E(X)^2 = 4/3 - 1 = 1/3$.

6. 8 fair dice are rolled. Let X be the number of dice that land on 6.

   (a) What is the expectation and variance of X?
   Solution: X is binomial with $n = 8$, $p = 1/6$, so $E(X) = np = 4/3$ and $\text{Var}(X) = np(1 - p) = 10/9$.

(b) What is the probability that X is at most 2?

Solution:

$$P(X \leq 2) = \sum_{i=0}^{2} \binom{8}{i} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{8-i} = .86...$$

(c) The first die that was rolled landed on 6. Given this information, what is now the probability that X is at most 2?

Solution: In order to have at most 2 sixes, we need to get at most one six on the next 7 rolls. This is a binomial calculation with $n = 7$ and $p = 1/6$, so the required probability is

$$\sum_{i=0}^{1} \binom{7}{i} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{7-i} = .66...$$

7. The St. Joseph County superior court has two court reporters, Mr. Jakes and Mr. Williams. On average, Mr. Jakes makes 2 typographic errors per page that he types, and Mr. Williams makes on average 5 errors per page.

(a) On a day when Mr. Jakes is working, I examine a randomly chosen page of the court report. What is the probability that I see at least 2 errors? (I am perfect at spotting errors.)

Solution: We model the number of mistakes as a Poisson random variable with $\lambda = 2$, so the probability of at least two errors is

$$1 - P(\text{zero errors} - P(1 \text{ error})) = 1 - e^{-2} - \frac{2^1}{1!}e^{-2} = 1 - 3e^{-2}.$$

(b) Mr. Jakes works three days a week, and Mr. Williams works the other two. On a randomly chosen day, I inspect a randomly chosen page of the court report. What is the probability that I find exactly 2 errors?

Solution: We model the number of mistakes the Mr. Jakes makes as a Poisson random variable with $\lambda = 2$, and the number the Mr. Williams makes as a Poisson with $\lambda = 5$. Let E be the event that I see exactly 2 mistakes, J the event that Mr. Jakes was working that day, and W the event that Mr. Williams was working that day. By the law of total probability we have

$$P(E) = P(E|J)P(J) + P(E|W)P(W) = \left(\frac{2^2}{2!}e^{-2}\right) * \left(\frac{3}{5}\right) + \left(\frac{5^2}{2!}e^{-5}\right) * \left(\frac{2}{5}\right) = \frac{6}{5e^2} + \frac{5}{e^5}.$$

Because Mr. Jakes and Mr. Williams make mistakes at different rates, it's not entirely correct to say that the average number of errors is $2 * (3/5) + 5 * (2/5) = 16/5$, and use a Poisson with $\lambda = 16/5$ to model the number of errors; by using a different Poisson random variable to model each of the two reporters, we get a more accurate answer.