Chapter 3

Problems

1. $P\{6 \mid \text{different}\} = P\{6, \text{different}\}/P\{\text{different}\}$

$$= \frac{P\{1\text{st} = 6, 2\text{nd} \neq 6\} + P\{1st \neq 6, 2\text{nd} = 6\}}{5/6}$$

$$= \frac{2\ 1/6\ 5/6}{5.6} = 1/3$$

could also have been solved by using reduced sample space—for given that outcomes differ it is the same as asking for the probability that 6 is chosen when 2 of the numbers 1, 2, 3, 4, 5, 6 are randomly chosen.

2. $P\{6 \mid \text{sum of } 7\} = P\{(6,1)\}/1/6 = 1/6$

$P\{6 \mid \text{sum of } 8\} = P\{(6,2)\}/5/36 = 1/5$

$P\{6 \mid \text{sum of } 9\} = P\{(6,3)\}/4/36 = 1/4$

$P\{6 \mid \text{sum of } 10\} = P\{(6,4)\}/3/36 = 1/3$

$P\{6 \mid \text{sum of } 11\} = P\{(6,5)\}/2/36 = 1/2$

$P\{6 \mid \text{sum of } 12\} = 1.$

3. $P\{E \text{ has } 3 \mid N-S \text{ has } 8\} = \dfrac{P\{E \text{ has } 3, N-S \text{ has } 8\}}{P\{N-S \text{ has } 8\}}$

$$= \frac{\binom{13}{8}\binom{39}{18}\binom{5}{3}\binom{21}{10}\Big/\binom{52}{26}\binom{26}{13}}{\binom{13}{8}\binom{39}{18}\Big/\binom{52}{26}} = .339$$

10. 11/50

12. (a) $(.9)(.8)(.7) = .504$

(b) Let F_i denote the event that she failed the ith exam.

$$P(F_2 \mid F_1^c\ F_2^c F_3^c)^c) = \frac{P(F_1^c F_2)}{1-.504} = \frac{(.9)(.2)}{.496} = .3629$$

18. (a) $P(\text{Ind} \mid \text{voted}) = \dfrac{P(\text{voted}|\text{Ind})P(\text{Ind})}{\sum P(\text{voted}|\text{type})P(\text{type})}$

$$= \frac{.35(.46)}{.35(.46) + .62(.3) + .58(.24)} \approx 331$$

(b) $P\{\text{Lib} \mid \text{voted}\} = \dfrac{.62(.30)}{.35(.46) + .62(.3) + .58(.24)} \approx .383$

(c) $P\{\text{Con} \mid \text{voted}\} = \dfrac{.58(.24)}{.35(.46) + .62(.3) + .58(.24)} \approx .286$

(d) $P\{\text{voted}\} = .35(.46) + .62(.3) + .58(.24) = .4862$
That is, 48.62 percent of the voters voted.

20. (a) $P(F \mid C) = \dfrac{P(FC)}{P(C)} = .02/.05 = .40$

(b) $P(C \mid F) = P(FC)/P(F) = .02/.52 = 1/26 \approx .038$

22. a. $\dfrac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \dfrac{5}{9}$

b. $\dfrac{1}{3!} = \dfrac{1}{6}$

c. $\dfrac{5}{9}\dfrac{1}{6} = \dfrac{5}{54}$

23. $P(w \mid w \text{ transferred}\}P\{w \text{ tr.}\} + P(w \mid R \text{ tr.}\}P\{R \text{ tr.}\} = \dfrac{2}{3}\dfrac{1}{3} + \dfrac{1}{3}\dfrac{2}{3} = \dfrac{4}{9}$.

$$P\{w \text{ transferred} \mid w\} = \frac{P\{w|w \text{ tr.}\}P\{w \text{ tr.}\}}{P\{w\}} = \frac{\frac{2}{3}\frac{1}{3}}{\frac{4}{9}} = 1/2.$$

24. (a) $P\{g - g \mid \text{at least one } g\} = \dfrac{1/4}{3/4} = 1/3.$

(b) Since we have no information about the ball in the urn, the answer is 1/2.

26. Let M be the event that the person is male, and let C be the event that he or she is color blind. Also, let p denote the proportion of the population that is male.

$$P(M \mid C) = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} = \frac{(.05)p}{(.05)p + (.0025)(1-p)}$$

28. Let A denote the event that the next card is the ace of spades and let B be the event that it is the two of clubs.

(a) $P\{A\} = P\{\text{next card is an ace}\}P\{A \mid \text{next card is an ace}\}$
$$= \frac{3}{32}\frac{1}{4} = \frac{3}{128}$$

(b) Let C be the event that the two of clubs appeared among the first 20 cards.

$$P(B) = P(B \mid C)P(C) + P(B \mid C^c)P(C^c)$$
$$= 0\frac{19}{48} + \frac{1}{32}\frac{29}{48} = \frac{29}{1536}$$

33. Let E and R be the events that Joe is early tomorrow and that it will rain tomorrow.

(a) $P(E) = P(E|R)P(R) + P(E|R^c)P(R^c) = .7(.7) + .9(.3) = .76$

(b) $P(R|E) = \dfrac{P(E|R)P(R)}{P(E)} = 49/76$

37. (a) $P\{\text{fair} \mid h\} = \dfrac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{3}$.

(b) $P\{\text{fair} \mid hh\} = \dfrac{\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{5}$.

(c) 1

53. Let W and F be the events that component 1 works and that the system functions.

$$P(W \mid F) = \frac{P(WF)}{P(F)} = \frac{P(W)}{1 - P(F^c)} = \frac{1/2}{1 - (1/2)^{n-1}}$$

57. (a) $2p(1-p)$

(b) $\binom{3}{2}p^2(1-p)$

(c) $P\{\text{up on first} \mid \text{up 1 after 3}\}$
$= P\{\text{up first, up 1 after 3}\}/[3p^2(1-p)]$
$= p2p(1-p)/[3p^2(1-p)] = 2/3.$

62. (a) $P\{\text{both hit} \mid \text{at least one hit}\} = \dfrac{P\{\text{both hit}\}}{P\{\text{at least one hit}\}}$

$$= p_1p_2/(1 - q_1q_2)$$

(b) $P\{\text{Barb hit} \mid \text{at least one hit}\} = p_1/(1 - q_1q_2)$
$Q_i = 1 - p_i$, and we have assumed that the outcomes of the shots are independent.

64. If use (a) will win with probability p. If use strategy (b) then

$$\begin{aligned} P\{\text{win}\} &= P\{\text{win} \mid \text{both correct}\}p^2 + P\{\text{win} \mid \text{exactly 1 correct}\}2p(1-p) \\ &\quad + P\{\text{win} \mid \text{neither correct}\}(1-p)^2 \\ &= p^2 + p(1-p) + 0 = p \end{aligned}$$

Thus, both strategies give the same probability of winning.

66. (a) $[I - (1 - P_1P_2)(1 - P_3P_4)]P_5 = (P_1P_2 + P_3P_4 - P_1P_2P_3P_4)P_5$

(b) Let $E_1 = \{1 \text{ and } 4 \text{ close}\}$, $E_2 = \{1, 3, 5 \text{ all close}\}$

$E_3 = \{2, 5 \text{ close}\}$, $E_4 = \{2, 3, 4 \text{ close}\}$. The desired probability is

71.
$$\begin{aligned} P\{\text{Braves win}\} &= P\{B \mid B \text{ wins 3 of 3}\}\ 1/8 + P\{B \mid B \text{ wins 2 of 3}\}\ 3/8 \\ &\quad + P\{B \mid B \text{ wins 1 of 3}\}\ 3/8 + P\{B \mid B \text{ wins 0 of 3}\}\ 1/8 \\ &= \frac{1}{8} + \frac{3}{8}\left[\frac{1}{4}\frac{1}{2} + \frac{3}{4}\right] + \frac{3}{8}\ \ \frac{3}{4}\frac{1}{2} = \frac{38}{64} \end{aligned}$$

where $P\{B \mid B \text{ wins } i \text{ of } 3\}$ is obtained by conditioning on the outcome of the other series. For instance

$$\begin{aligned} P\{B \mid B \text{ win 2 of 3}\} &= P\{B \mid D \text{ or } G \text{ win 3 of 3}, B \text{ win 2 of 3}\}\ 1/4 \\ &= P\{B \mid D \text{ or } G \text{ win 2 of 3}, B \text{ win 2 of 3}\}\ 3/4 \\ &= \frac{1}{2}\frac{1}{4} + \frac{3}{4}. \end{aligned}$$

By symmetry $P\{D \text{ win}\} = P\{G \text{ win}\}$ and as the probabilities must sum to 1 we have.

$$P\{D \text{ win}\} = P\{G \text{ win}\} = \frac{13}{64}.$$

73. (a) 1/16, (b) 1/32, (c) 10/32, (d) 1/4, (e) 31/32.

78. (a) Because there will be 4 games if each player wins one of the first two games and then one of them wins the next two, $P(4 \text{ games}) = 2p(1-p)[p^2 + (1-p)^2]$.

(b) Let A be the event that A wins. Conditioning on the outcome of the first two games gives

$$P(A = P(A|a, a)p^2 + P(A|a, b)p(1-p) + P(A|b, a)(1-p)p + P(A|b, b)(1-p)^2$$
$$= p^2 + P(A)2p(1-p)$$

where the notation a, b means, for instance, that A wins the first and B wins the second game. The final equation used that $P(A|a, b) = P(A|b, a) = P(A)$. Solving, gives

$$P(A) = \frac{p^2}{1-2p(1-p)}$$

86. Using the hint

$$P\{A \subset B\} = \sum_{i=0}^{n}(2^i/2^n)\binom{n}{i}\Big/2^n = \sum_{i=0}^{n}\binom{n}{i}2^i/4^n = (3/4)^n$$

where the final equality uses

$$\sum_{i=0}^{n}\binom{n}{i}2^i 1^{n-i} = (2+1)^n$$

(b) $P(AB = \phi) = P(A \subset B^c) = (3/4)^n$, by part (a), since B^c is also equally likely to be any of the subsets.

Theoretical Exercises

1. $$P(AB|A) = \frac{P(AB)}{P(A)} \geq \frac{P(AB)}{P(A \cup B)} = P(AB|A \cup B)$$

2. If $A \subset B$

$$P(A|B) = \frac{P(A)}{P(B)}, \quad P(A|B^c) = 0, \quad P(B|A) = 1, \quad P(B|A^c) = \frac{P(BA^c)}{P(A^c)}$$

5) a) $F \searrow E \Rightarrow P(E|F) \leq P(E)$

$\Rightarrow \frac{P(EF)}{P(F)} \leq P(E)$

$\Rightarrow P(EF) \leq P(E)P(F)$

$\Rightarrow P(EF)/P(E) \leq P(F)$

$\Rightarrow P(F|E) \leq P(F)$

$\Rightarrow E \searrow F.$

So true

b) Not always true

(Example: $F = G$, E and F independent (so E and G too)

$P(F) < 1$ (so $P(G) < 1$)

Then $F \searrow E$ and $E \searrow G$,

But $P(G|F) = 1 \nleq P(G)$)

c) Not always true

~~(Example: $F = G^c$, E ind. of F (so also G)~~

~~Then $F \searrow E$, $G \searrow E$,~~

~~But $FG = \phi$, and $P(E|\phi)$~~

(Example: rolling a dice

$E = \{1, 2, 3\}$

$F = \{3, 4\}$

$G = \{3, 5\}$

Check: $F \searrow E$, $G \searrow E$ (with equality)

But $P(E|FG) = 1 \nleq P(E)$)

For $\nearrow$: a) True (by same argument)

b) Same as before, but now take $F = G^c$, $0 < P(F) < 1$

If E ind. of F, then

$F \nearrow E$, $E \nearrow G$, but $FG = \phi$,

~~So $P(\phi|F) = 0 \ngtr$~~

So $P(G|F) = 0 \ngtr P(G)$

c) Same as before, but take

$G = \{2, 4\}$

Check: $F \nearrow E$, $G \nearrow E$,

but $FG = \{4\}$,

and so $P(E|FG) = 0 \neq P(E)$

6. $P\left(\bigcup_1^n E_i\right) = 1 - P\left(\bigcap_1^n E_i^c\right) = 1 - \prod_1^n [1 - P(E_i)]$

10. $P(A_{i,j}) = 1/365$. For $i \neq j \neq k$, $P(A_{i,j}A_{j,k}) = 365/(365)^3 = 1/(365)^2$. Also, for $i \neq j \neq k \neq r$, $P(A_{i,j}A_{k,r}) = 1/(365)^2$.

16. If the first trial is a success, then the remaining $n - 1$ must result in an odd number of successes, whereas if it is a failure, then the remaining $n - 1$ must result in an even number of successes.