Chapter 5

Problems

1. (a) $c\int_{-1}^{1}(1-x^2)dx = 1 \Rightarrow c = 3/4$

(b) $F(x) = \frac{3}{4}\int_{-1}^{x}(1-x^2)dx = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right), -1 < x < 1$

3. No. $f(5/2) < 0$

5. Must choose c so that

$$.01 = \int_{c}^{1} 5(1-x)^4 dx = (1-c)^5$$

so $c = 1 - (.01)^{1/5}$.

6. (a) $E[X] = \frac{1}{4}\int_{0}^{\infty} x^2 e^{-x/2}dx = 2\int_{0}^{\infty} y^2 e^{-y}dx = 2\Gamma(3) = 4$

(b) By symmetry of $f(x)$ about $x = 0$, $E[X] = 0$

(c) $E[X] = \int_{5}^{\infty}\frac{5}{x}dx = \infty$

7. $\int_{0}^{1}(a + bx^2)dx = 1$ or $a + \frac{b}{3} = 1$

$\int_{0}^{1}x(a+bx^2)dx = \frac{3}{5}$ or $\frac{a}{2} + \frac{b}{4} = 3/5$. Hence,

$a = \frac{3}{5}$, $b = \frac{6}{5}$

10. (a) $P\{\text{goes to } A\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60\}$.
$= 2/3$ since X is uniform (0, 60).

(b) same answer as in (a).

11. X is uniform on $(0, L)$.

$$P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < 1/4\right\}$$
$$= 1 - P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) > 1/4\right\}$$
$$= 1 - P\left\{\frac{X}{L-X} > 1/4, \frac{L-X}{X} > 1/4\right\}$$
$$= 1 - P\{X > L/5, X < 4L/5\}$$
$$= 1 - P\left\{\frac{L}{5} < X < 4L/5\right\}$$
$$= 1 - \frac{3}{5} = \frac{2}{5}.$$

14. $E[X^n] = \int_0^1 x^n dx = \dfrac{1}{n+1}$

$P\{X^n \le x\} = P\{X \le x^{1/n}\} = x^{1/n}$

$E[X^n] = \int_0^1 x \frac{1}{n} x^{\left(\frac{1}{n}-1\right)} dx = \frac{1}{n}\int_0^1 x^{1/n} dx = \dfrac{1}{n+1}$

15. (a) $\Phi(.8333) = .7977$
(b) $2\Phi(1) - 1 = .6827$
(c) $1 - \Phi(.3333) = .3695$
(d) $\Phi(1.6667) = .9522$
(e) $1 - \Phi(1) = .1587$

17. $E[\text{Points}] = 10(1/10) + 5(2/10) + 3(2/10) = 2.6$

19. Letting $Z = (X - 12)/2$ then Z is a standard normal. Now, $.10 = P\{Z > (c - 12)/2\}$. But from Table 5.1, $P\{Z < 1.28\} = .90$ and so

$(c - 12)/2 = 1.28$ or $c = 14.56$

20. Let X denote the number in favor. Then X is binomial with mean 65 and standard deviation $\sqrt{65(.35)} \approx 4.77$. Also let Z be a standard normal random variable.

(a) $P\{X \ge 50\} = P\{X \ge 49.5\} = P\{X - 65\}/4.77 \ge -15.5/4.77$
$\approx P\{Z \ge -3.25\} \approx .9994$

(b) $P\{59.5 \le X \le 70.5\} \approx P\{-5.5/4.77 \le Z \le 5.5/4.77\}$
$= 2P\{Z \le 1.15\} - 1 \approx .75$

(c) $P\{X \le 74.5\} \approx P\{Z \le 9.5/4.77\} \approx .977$

21: X = height of randomly chosen man
X = Normal $(71, 6.25)$

$P(X \geq 74) = P\left(\frac{X-71}{\sqrt{6.25}} \geq \frac{74-71}{\sqrt{6.25}}\right) = P(Z \geq 1.2) = 1 - .8849$
$= .1151$

$P(X \geq 77 \mid X \geq 72) = \dfrac{P(X \geq 77)}{P(X \geq 72)} = \dfrac{P(Z \geq 2.4)}{P(Z \geq .4)} = .0238$

22. (a) $P\{.9000 - .005 < X < .9000 + .005\}$

$$= P\left\{-\frac{.005}{.003} < Z < \frac{.005}{.003}\right\}$$
$$= P\{-1.67 < Z < 1.67\}$$
$$= 2\Phi(1.67) - 1 = .9050.$$

Hence 9.5 percent will be defective (that is each will be defective with probability $1 - .9050 = .0950$).

(b) $P\left\{-\frac{.005}{\sigma} < Z < \frac{.005}{\sigma}\right\} = 2\Phi\left(\frac{.005}{\sigma}\right) - 1 = .99$ when

$$\Phi\left(\frac{.005}{\sigma}\right) = .995 \Rightarrow \frac{.005}{\sigma} = 2.575 \Rightarrow \sigma = .0019.$$

23. (a) $P\{149.5 < X < 200.5\} = P\left\{\frac{149.5 - \frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}} < Z < \frac{200.5 - \frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}}\right\}$

$$= \Phi\left(\frac{200.5 - 166.7}{\sqrt{5000/36}}\right) - \Phi\left(\frac{149.5 - 166.7}{\sqrt{5000/36}}\right)$$
$$\approx \Phi(2.87) + \Phi(1.46) - 1 = .9258.$$

(b) $P\{X < 149.5\} = P\left\{Z < \frac{149.5 - 800(1/5)}{\sqrt{800\frac{1}{5}\frac{4}{5}}}\right\}$

$$= P\{Z < -.93\}$$
$$= 1 - \Phi(.93) = .1762.$$

26:

P(Reaching False Conclusion | Coin is fair)

$= P(\geq 525 \text{ heads} \mid \text{Coin fair})$

$= P(\text{Binomial}(1000, \tfrac{1}{2}) \geq 525)$

$\approx P(\text{Normal}(500, 250) \geq 525)$

$= P\left(Z \geq \frac{525 - 500}{\sqrt{250}} = 1.58\right) = .0571$

P(False conclusion | Coin biased)

$= P(\leq 525 \text{ heads} \mid \text{Coin biased})$

$= P(\text{Binomial}(1000, .55) \leq 525)$

$\approx P\left(Z \leq \frac{525 - 550}{\sqrt{1000 \times .55 \times .45}} = -1.59\right) = .0559$

31. (a) $E[|X-a|] = \int_a^A (x-a)\frac{dx}{A} + \int_0^a (a-x)\frac{dx}{A} = \frac{A}{2} - \left(a - \frac{a^2}{A}\right)$

$$\frac{d}{da}(\ ) = \frac{2a}{A} - 1 = 0 \Rightarrow a = A/2$$

(b) $E[|X-a|] = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx$

$$= a(1-e^{-\lambda a}) + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - ae^{-\lambda a}$$

Differentiation yields that the minimum is attained at $\bar{a}$ where

$$e^{-\lambda \bar{a}} = 1/2 \text{ or } \bar{a} = \log 2/\lambda$$

(c) Minimizing a = median of F

34. (a) $P\{X > 20\} = e^{-1}$

(b) $P\{X > 30 | X > 10\} = \frac{P\{X > 30\}}{P\{X > 10\}} = \frac{1/4}{3/4} = 1/3$

Theoretical Exercises

6. Let X be uniform on (0, 1) and define E_a to be the event that X is unequal to a. Since $\bigcap_a E_a$ is the empty set, it must have probability 0.

8. Since $0 \le X \le c$, it follows that $X^2 \le cX$. Hence,

$$\begin{aligned}
\text{Var}(X) &= E[X^2] - (E[X])^2 \\
&\le E[cX - (E[X])^2 \\
&= cE[X] - (E[X])^2 \\
&= E[X](c - E[X]) \\
&= c^2[\alpha(1-\alpha)] \quad \text{where } \alpha = E[X]/c \\
&\le c^2/4
\end{aligned}$$

where the last inequality first uses the hypothesis that $P\{0 \le X \le c\} = 1$ to calculate that $0 \le \alpha \le 1$ and then uses calculus to show that $\underset{0\le\alpha\le1}{\text{maximum}}\ \alpha(1-\alpha) = 1/4$.