CHAPTER 6

1) a)

X \ Y	2	3	4	5	6	7	8	9	10	11	12
1	1										
2		2	1								
3			2	2	1						
4				2	2	2	1				
5					2	2	2	2	1		
6						2	2	2	2	2	1

$P(X=x, Y=y) = \dfrac{xy \text{ entry above}}{36}$ (empty entries are 0)

b)

X \ Y	1	2	3	4	5	6
1	1	2	2	2	2	2
2		1	2	2	2	2
3			1	2	2	2
4				1	2	2
5					1	2
6						1

$P(X=x, Y=y) = \dfrac{xy \text{ entry}}{36}$ (empty entries 0)

6)

$N_1 \backslash N_2$	1	2	3	4	5
1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0
2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0
3	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0
4	$\frac{1}{10}$	0	0	0	0
5	0	0	0	0	0

120 possible orderings of the 5 transistors

$$P(N_1 = 1, N_2 = 1) = P(\underset{\substack{\uparrow \\ \text{defective}}}{D}DGG\underset{\substack{\uparrow \\ \text{Good}}}{G}) = \frac{2!\,3!}{120} = \frac{1}{10}$$

$$P(N_1 = 1, N_2 = 2) = P(DGDGG) = \frac{2 \times 3 \times 1 \times 2 \times 1}{120} = \frac{1}{10}$$

etc.

8. $f_Y(y) = c\int_{-y}^{y} (y^2 - x^2)e^{-y}dx$

$= \frac{4}{3}cy^3e^{-y}, -0 < y < \infty$

$\int_0^\infty f_Y(y)dy = 1 \Rightarrow c = 1/8$ and so $f_Y(y) = \frac{y^3e^{-y}}{6}, 0 < y < \infty$

$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}(y^2 - x^2)e^{-y}dy$

$= \frac{1}{4}e^{-|x|}(1+|x|)$ upon using $-\int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$

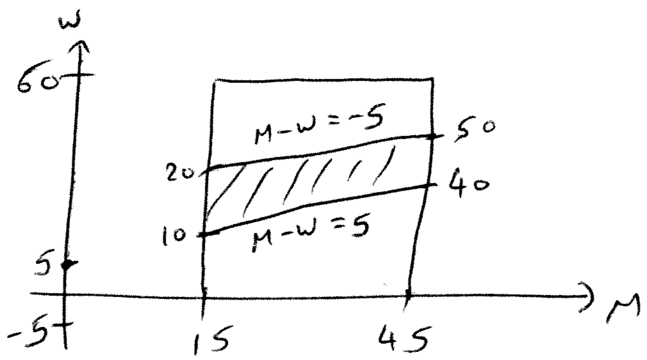
9. (b) $f_X(x) = \frac{6}{7}\int_0^2\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}(2x^2 + x)$

(c) $P\{X > Y\} = \frac{6}{7}\int_0^1\int_0^x\left(x^2 + \frac{xy}{2}dydx\right) = \frac{15}{56}$

(d) $P\{Y > 1/2 \mid X < 1/2\} = P\{Y > 1/2, X < 1/2\}/P\{X < 1/2\}$

$$= \frac{\int_{1/2}^{2}\int_0^{1/2}\left(x^2 + \frac{xy}{2}dxdy\right)}{\int_0^{1/2}(2x^2 + x)dx}$$

13: M = arrival time of man, uniform on [15, 45]
W = " " " woman, " " [0, 60]



Joint density of (M, w):

$\frac{1}{30 \times 60}$ on $[15, 45] \times [0, 60]$,

0 elsewhere

P(first to arrive waits ≤ 5 minutes)

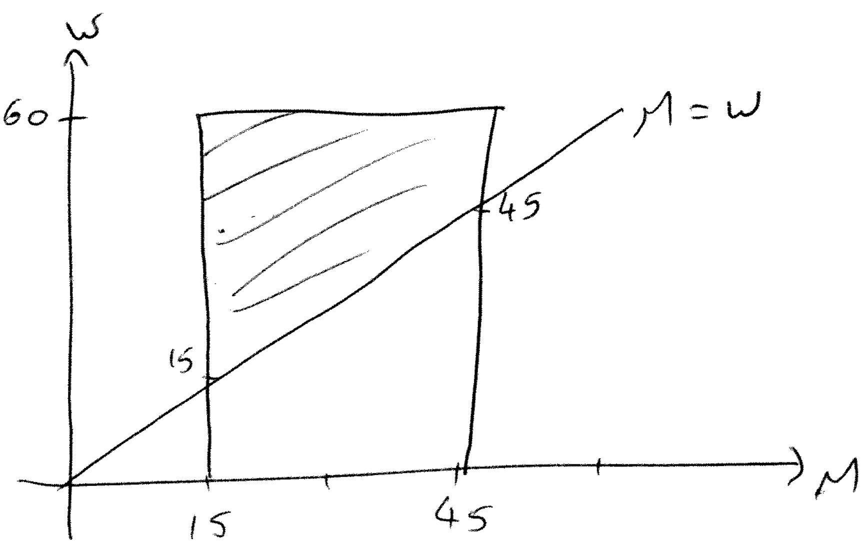
$= P(|M-W| \leq 5)$

$= P(-5 \leq M - W \leq 5)$

$= \dfrac{\text{Shaded area on previous page}}{30 \times 60}$

$= \dfrac{10 \times 30}{30 \times 60} = \dfrac{1}{6}$

P(Man arrives first) $= P(M < W)$



$= \dfrac{\text{Shaded area above}}{30 \times 60} = \dfrac{1}{2}.$

17. $\frac{1}{3}$ since each of the 3 points is equally likely to be the middle one.

19. $\int_0^1 \int_0^x \frac{1}{x}\, dydx = \int_0^1 dx = 1$

(a) $\int_y^1 \frac{1}{x} dx = -\ln(y)$, $0 < y < 1$

(b) $\int_0^x \frac{1}{x} dy = 1$, $0 < y < 1$

(c) $\frac{1}{2}$

(d) Integrating by parts gives that

$$\int_0^1 y\ \ln(y)dy = -1 - \int_0^1 (y\ln(y) - y)dy$$

yielding the result

$$E[Y] = -\int_0^1 y\ln(y)dy = 1/4$$

20. (a) yes: $f_X(x) = xe^{-x}$, $f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

(b) no: $f_X(x) = \int_x^1 f(x,y)dy = 2(1-x)$, $0 < x < 1$

$$f_Y(y) = \int_0^y f(x,y)dx = 2y,\ 0 < y < 1$$

22. (a) No, since the joint density does not factor.

(b) $f_X(x) = \int_0^1 (x+y)dy = x + 1/2$, $0 < x < 1$.

(c) $P\{X + Y < 1\} = \int_0^1 \int_0^{1-x} (x+y)dydx$

$$= \int_0^1 [x(1-x) + (1-x)^2/2]dx = 1/3$$

27. $P\{X_1/X_2 < a\} = \int_0^\infty \int_0^{ay} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dxdy$

$$= \int_0^\infty \left(1 - e^{-\lambda_1 ay}\right)\lambda_2 e^{-\lambda_2 y} dy$$

$$= 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 a} = \frac{\lambda_1 a}{a\lambda_1 + \lambda_2}$$

$$P\{X_1/X_2 < 1\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

THEORETICAL EXERCISES

5. (a) For $a > 0$

$$F_Z(a) = P\{X \le aY\}$$
$$= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)dxdy$$
$$= \int_0^{\infty} F_X(ay) f_Y(y)dy$$
$$f_Z(a) = \int_0^{\infty} f_X(ay)yf_Y(y)dy$$

(b)

$$F_Z(a) = P\{XY < a\}$$
$$= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)dxdy$$
$$= \int_0^{\infty} F_X(a/y) f_Y(y)dy$$
$$f_Z(a) = \int_0^{\infty} f_X(a/y)\frac{1}{y} f_Y(y)dy$$

If X is exponential with rate λ and Y is exponential with rate μ then (a) and (b) reduce to

(a) $F_Z(a) = \int_0^{\lambda} \lambda e^{-\lambda ay} y\mu e^{-\mu y}dy$

(b) $F_Z(a) = \int_0^{\infty} \lambda e^{-\lambda a/y} \frac{1}{y} \mu e^{-\mu y}dy$

9. $P\{\min(X_1, \ldots, X_n) > t\} = P\{X_1 > t, \ldots, X_n > t\}$
$= e^{-\lambda t} \ldots e^{-\lambda t} = e^{-n\lambda t}$

thus showing that the minimum is exponential with rate $n\lambda$.